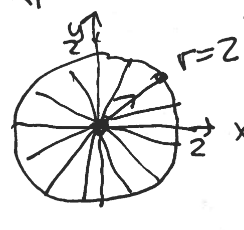
MAT2500-01/04 19S Final Exam Answers

(1) $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} \underbrace{\langle x^2y, -xy^2\rangle}_{\vec{F}(x,y)} \cdot \underbrace{\langle dx, dy\rangle}_{d\vec{r}}$

a) $\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y} = \frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)$

$= -y^2 - x^2 = -r^2$

$\iint_{R_1} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \int_0^{2\pi}\int_0^2 (-r^2)\, r\,dr\,d\theta$



$= -\int_0^{2\pi} d\theta \int_0^2 r^3\, dr$

$= -\left(\theta\Big|_0^{2\pi}\right)\left(\frac{r^4}{4}\Big|_0^2\right)$

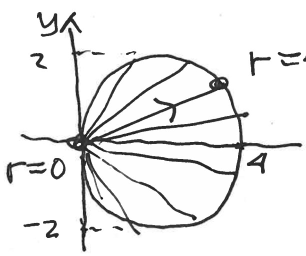
$R: 0 \le r \le 2$

$0 \le \theta \le 2\pi$

$= -2\pi\cdot\frac{2^4}{4} = \boxed{-8\pi}$

b) $x^2+y^2 = 4x \to r^2 = 4r\cos\theta$

$\hookrightarrow r = 4\cos\theta$



$R_2: -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 4\cos\theta$

$\iint_{R_2} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{4\cos\theta} (-r^2)\, r\,dr\,d\theta$

$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{4\cos\theta} r^3\,dr\,d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_{r=0}^{r=4\cos\theta} d\theta$

$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4^4}{4}\cos^4\theta\, d\theta = -64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta$

Maple $=$ $\boxed{-24\pi}$

c) $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle \underbrace{r(t)}_{4\cos t}\cos(\underbrace{\theta(t)}_{t}), r(t)\sin\theta(t)\rangle$

$= \langle (4\cos t)\cos t, (4\cos t)\sin t\rangle =$

$= 4\langle \cos^2 t, \cos t\sin t\rangle$

$\vec{r}'(t) = 4\langle -2\cos t\sin t, -\sin^2 t + \cos^2 t\rangle$

c) continued.

$F(\vec{r}(t)) = \langle (4\cos^2 t)^2\, 4\cos t\sin t, -(4\cos^2 t)(4\cos t\sin t)^2\rangle$

$= 4^3\langle \cos^5 t\sin t, -\cos^4 t\sin^2 t\rangle$

$\vec{F}(r(t))\cdot\vec{r}'(t) = 4^4[(\cos^5 t\sin t)(-2\cos t\sin t) + (-\cos^4 t\sin^2 t)(-\sin^2 t + \cos^2 t)]$

$= 4^4[-2\cos^6 t\sin^2 t + \cos^4 t\sin^4 t - \cos^6 t\sin^2 t]$

$\oint_{C_2} \vec{F}\cdot d\vec{r} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^4(\cdots)\,dt$

Maple $=$ $\boxed{-24\pi}$ agrees!

(2) $\vec{r}(t) = \langle \cos(t), \sin t, \sin t\rangle = \langle x, y, z\rangle$

$\vec{r}'(t) = \langle -\sin t, \cos t, \cos t\rangle$

$\vec{F}(\vec{r}(t)) = \langle x^2y, yz, zx^2\rangle\Big|_{\vec{r}=\vec{r}(t)}$

$= \langle (\cos t)^2\sin t, (\sin t)(\sin t), \sin t(\cos t)^2\rangle$

$= \langle \cos^2 t\sin t, \sin^2 t, \cos^2 t\sin t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = \cos^2 t\sin t(-\sin t) + \sin^2 t(\cos t) + \cos^2 t\sin t(\cos t)$

$= -\cos^2 t\sin^2 t + \cos t\sin^2 t + \cos^3 t\sin t.$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}(-\cos^2 t\sin^2 t + \cos t\sin^2 t + \cos^3 t\sin t)\,dt$

Maple $=$ $\boxed{-\frac{\pi}{4}}$

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(3) $\vec{F} = \langle F_1, F_2, F_3 \rangle = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z \rangle$

a) $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= \frac{\partial}{\partial x}(3x^2y - 3y) + \frac{\partial}{\partial y}(x^3z - 3x) + \frac{\partial}{\partial z}(x^3y + 2z)$

$= 6xyz + 0 + 2 = 6xyz + 2$

b) $\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \langle \frac{\partial}{\partial y}(x^3y + 2z) - \frac{\partial}{\partial z}(x^3z - 3x), \frac{\partial}{\partial z}(3x^2yz - 3y) - \frac{\partial}{\partial x}(x^3y + 2z),$

$\frac{\partial}{\partial x}(x^3z - 3x) - \frac{\partial}{\partial y}(3x^2yz - 3y) \rangle$

$= \langle x^3 - x^3, 3x^2y - 3x^2y, 3x^2z - 3 - 3x^2z + 3 \rangle$

$= \langle 0, 0, 0 \rangle = \vec{0}$ ✓

c) $\vec{F} = \vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$:

$\int \left[\frac{\partial f}{\partial x} = 3x^2yz - 3y \right] dx \to f = \int 3x^2yz - 3y \, dx = x^3yz - 3xy + C(y,z)$

$\frac{\partial f}{\partial y} = x^3z - 3x \longleftarrow \frac{\partial f}{\partial y} = x^3z - 3x + \frac{\partial C(y,z)}{\partial y} = x^3z - 3x$

$\frac{\partial C(y,z)}{\partial y} = 0 \to C(y,z) = C(z)$

$\frac{\partial f}{\partial z} = x^3y + 2z$

$f = x^3yz - 3xy + C(z)$

$\frac{\partial f}{\partial z} = x^3y + C'(z) = x^3y + 2z$

$\int [C'(z) = 2z] \, dz$

$C(z) = \int 2z \, dz = z^2 + k$

$\boxed{f = x^3yz - 3xy + z^2 + k}$

Check: $\vec{\nabla} f = \vec{F}$ ✓

d) set $k = 0$:

$\int_C \vec{F} \cdot d\vec{r} = f(2,3,4) - f(1,2,3)$

$= 2^3 3 \cdot 4 - 3 \cdot 2 \cdot 3 + 4^2 - (1^3 \cdot 2 \cdot 3 - 3 \cdot 1 \cdot 2 + 3^2)$

$\overset{\text{Maple}}{=} \boxed{85}$

e) $\vec{r} = r_1 + t(\vec{r_2} - \vec{r_1}) = \langle 1,2,3 \rangle + t(\langle 2,3,4 \rangle - \langle 1,2,3 \rangle)$

$= \langle 1,2,3 \rangle + t\langle 1,1,1 \rangle$

$= \langle 1+t, 2+t, 3+t \rangle$

$\vec{r}'(t) = \langle 1,1,1 \rangle$

$\vec{F}(\vec{r}(t)) = \langle 3(1+t)^2(2+t)(3+t) - 3(2+t), (1+t)^3(3+t) - 3(1+t), (1+t)^3(2+t) + 2(3+t) \rangle$

$F(\vec{r}(t)) \cdot \vec{r}'(t) =$ sum of 3 components above

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (\ldots) \, dt \overset{\text{Maple}}{=} \boxed{85}$ ✓ checks!

1985 was our first championship!