MAT2500-01/04 18S Quiz 7   Print Name (Last, First) ____________________________|___

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

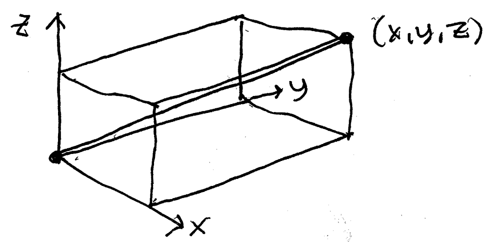
If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

Although the result may be obvious from symmetry considerations, respond to this max-min problem as follows. Draw a box indicating the three dimensions x, y, z and indicate their allowed values. Identify the volume function V and use the given constraint to eliminate z and express V as a function $f(x, y)$ and draw a graph of the bounded region of the plane over which this function is positive. Find the single critical point (x_0, y_0) in the interior of this region and locate it in your diagram [hint: subtract your two first derivative equations to get started in solving them] and then evaluate the volume $f(x_0, y_0)$ there. This must be the maximum value. Answer the word problem with a complete English sentence describing completely your result.

Optional.

Use Maple to evaluate second derivatives at this critical point to confirm that this is a local maximum of the function. This can even be done by hand.

▶ solution



$x, y, z > 0$ physical dimensions

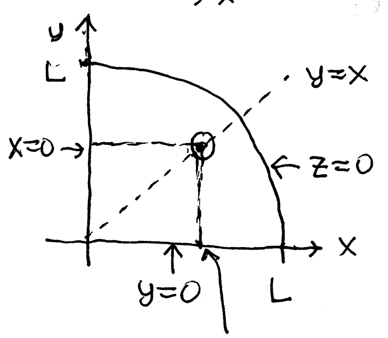
Diagonal of box: $\sqrt{x^2+y^2+z^2} = L$ (constraint)

Volume of box: $V = xyz$ (function to be maximized)

use to eliminate z: $z = \sqrt{L^2-x^2-y^2} > 0$

so $x^2+y^2 < L^2$, $x>0$, $y>0$ interior of quarter circle of radius L in first quadrant

$V > 0$ inside,
$V = 0$ on boundary
so must have a maximum in the interior.



side: $\frac{L}{\sqrt{3}} \approx 0.577L$

$$V = xy(L^2-x^2-y^2) \equiv f(x,y)$$

$$f_x(x,y) \overset{\text{Maple}}{=} \frac{y(L^2-2x^2-y^2)}{(L^2-x^2-y^2)^{1/2}} = 0 \longrightarrow 2x^2+y^2 = L^2$$

$$f_y(x,y) \overset{\text{Maple}}{=} \frac{x(L^2-x^2-2y^2)}{(L^2-x^2-y^2)^{1/2}} = 0 \longrightarrow x^2+2y^2 = L^2$$

$$-: \quad x^2-y^2 = 0 \;\to\; y = x$$

$$3x^2 = L^2$$

$$x = L/\sqrt{3} = y$$

is a cube of side $L/\sqrt{3}$ $\leftarrow$ $z = \left(L^2 - \frac{L^2}{3} - \frac{L^2}{3}\right)^{1/2} = \frac{L}{\sqrt{3}}$

$$V = f\left(\tfrac{L}{\sqrt{3}}, \tfrac{L}{\sqrt{3}}\right) = \left(\tfrac{L}{\sqrt{3}}\right)^3 = \frac{L^3}{3\sqrt{3}}$$

$\boxed{\text{The largest possible volume is } \frac{L^3}{3\sqrt{3}} \text{ for a cube of side } \frac{L}{\sqrt{3}}.}$ ← summary of complete findings

(see reverse for hand derived derivatives and optional part) partial derivatives by hand

$f(x,y) = xy\,(L^2-x^2-y^2)^{1/2}$

$f_x(x,y) = y\,(L^2-x^2-y^2)^{1/2} + xy\left(\tfrac12\right)(L^2-x^2-y^2)^{-1/2}(-2x)$

$= \dfrac{y(L^2-x^2-y^2) - yx^2}{(L^2-x^2-y^2)^{1/2}} = \dfrac{y(L^2-2x^2-y^2)}{(L^2-x^2-y^2)^{1/2}}$

just interchange (x,y) to get y partial derivative

$f_y(x,y) = x\,(L^2-x^2-y^2)^{1/2} + xy\left(\tfrac12\right)(L^2-x^2-y^2)^{-1/2}(-2y)$

$= \dfrac{x(L^2-x^2-y^2) - xy^2}{(L^2-x^2-y^2)^{1/2}} = \dfrac{x(L^2-x^2-2y^2)}{(L^2-x^2-y^2)^{1/2}}$

$$f_{xx}(x,y) = y\,\frac{(L^2-x^2-y^2)^{1/2}\,[\,-4x\,] - (L^2-x^2-2y^2)\left(\tfrac12\right)(L^2-x^2-y^2)^{-1/2}(-2x)}{(L^2-x^2-y^2)}$$

$$= y\,\frac{(L^2-x^2-y^2)(-4x) + (L^2-x^2-2y^2)\,x}{(L^2-x^2-y^2)^{3/2}} = \frac{xy\,[-4L^2+4x^2+4y^2+L^2-x^2-2y^2]}{(L^2-x^2-y^2)^{3/2}}$$

$$= \frac{xy\,[-3L^2+3x^2+2y^2]}{(L^2-x^2-y^2)^{3/2}}$$

interchange (x,y)

$$f_{yy}(x,y) = \frac{xy\,(-3L^2+2x^2+3y^2)}{(L^2-x^2-y^2)^{3/2}}$$

$$f_{xy}(x,y) = \frac{(L^2-x^2-y^2)^{1/2}\,[(L^2-2x^2-y^2)-2y^2] - y\,[L^2-2x^2-y^2]\left[\tfrac12\right](L^2-x^2-y^2)^{-1/2}(-2y)}{(L^2-x^2-y^2)}$$

$$= \frac{(L^2-x^2-y^2)(L^2-2x^2-3y^2) + y^2(L^2-2x^2-y^2)}{(L^2-x^2-y^2)^{3/2}}$$

Let $s = \frac{L}{\sqrt3} = x_0 = y_0 = z_0$

$f_{xx}(s,s) = \dfrac{s^2(-3L^2+5s^2)}{s^3} = \dfrac{\frac53 L^2 - 3L^2}{s} = \dfrac{-4L^2}{3s} = -\dfrac{4L^2\sqrt3}{3L} = -\dfrac{4\sqrt3}{3}L < 0 \Rightarrow$ local max

$f_{yy}(s,s) = \dots -\dfrac{4\sqrt3}{3}L < 0 \Rightarrow$ local max

$f_{xy}(s,s) = \dfrac{(L^2-\frac23 L^2)(L^2-\frac53 L^2) + \frac{L^2}{3}\cancelto{0}{(L^2-L^2)}}{s^3} = \dfrac{\left(\frac{L^2}{3}\right)\left(-\frac23 L^2\right)}{s^3} = -\dfrac29\,\dfrac{L^4\,3\sqrt3}{L^3} = -\dfrac{2\sqrt3}{3}L$

$f_{xx}(s,s)\,f_{yy}(s,s) - f_{xy}(s,s)^2 = \left(-\dfrac{4\sqrt3}{3}L\right)^2 - \left(-\dfrac{2\sqrt3}{3}L\right)^2 = (4^2-2^2)\dfrac{L^2}{3} = 4L^2 > 0$

confirms local max. along coord directions

although this was tedious, there was nothing learned from it compared to letting Maple supply the results. Maple is the appropriate tool to use here.