MAT 2500-01/04 18S Final Exam Answers (1)

1. a) b) $x^2+(y+2)^2=4$: $C(0,-2)$, radius $=2$

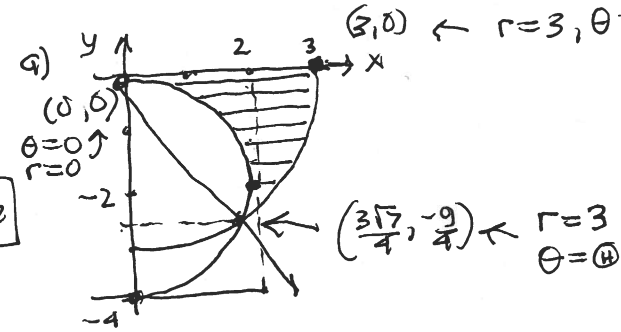
$x^2+y^2=9$ $\quad$ $C(0,0)$, radius $=3$

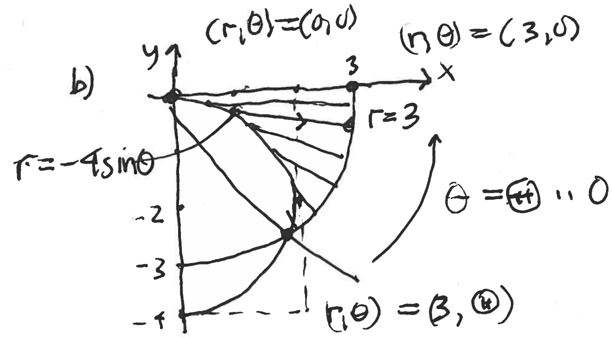
$\underbrace{x^2+y^2}_{r^2}+\underbrace{4y}_{r\sin\theta}+\cancel{4=4}\,0$ $\quad$ $r^2=9$ $\quad$ $\boxed{r=3}$ $\quad$ $\boxed{x=\sqrt{9-x^2}}$

$r^2+4r\sin\theta=0$ $\quad$ $x^2=-4y-y^2$

$r+4\sin\theta=0$ $\quad$ $\boxed{x=\sqrt{-4y-y^2}}$

$\boxed{r=-4\sin\theta}$





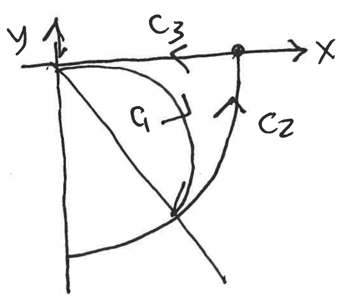
intersection: $\{\underbrace{x^2+y^2}_{=9}+4y=0,\ x^2+y^2=9\}$

$y=-\frac{9}{4} \rightarrow x=\sqrt{9-\left(\frac{9}{4}\right)^2}=3\sqrt{1-\frac{9}{16}}=\frac{3\sqrt7}{4}$

or $\{r=3,\ r=-4\sin\theta\} \rightarrow \sin\theta=-\frac{3}{4}$

$\theta=\boxed{-\arcsin\frac{3}{4}=-\theta_0 \equiv 4 \quad \approx -49°}$

4, 3, θ_0, $\sqrt7=\sqrt{16-9}$ $\quad$ $\theta_0=\arcsin\frac{3}{4}\boxed{=\arctan\frac{3}{\sqrt7}}\approx 49°$



c) $\vec F=\langle -y,2x\rangle$ $\quad$ $\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=\frac{\partial}{\partial x}(2x)-\frac{\partial}{\partial y}(-y)=2+1=3$

$$\iint_R \underbrace{\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}}_{3}\,dA=\boxed{\int_4^0\int_{-4\sin\theta}^3 (3)\underbrace{r\,dr\,d\theta}_{dA}}$$

$$\overset{\text{maple}}{=}\boxed{\frac{3}{2}\arctan\frac{3}{\sqrt7}+\frac{9\sqrt7}{4}\approx 7.2258}$$

C_3: $x=t,\ y=0$. $\vec r(t)=\langle t,0\rangle$ $t=3..0$ $\quad$ $\vec r\,'(t)=\langle 1,0\rangle$

$F(\vec r(t))=\langle (0),2t\rangle$ $\longrightarrow$ $\vec F(r(t))\cdot\vec r\,'(t)=0$

$\boxed{\int_{C_3}\vec F\cdot dr=0}$

C_2: $y=t,\ x=\sqrt{9-t^2}$ $\quad$ $\vec r(t)=\langle\sqrt{9-t^2},t\rangle$, $t=-\frac{9}{4}..0$ $\quad$ $\vec r\,'(t)=\left\langle\frac{-t}{\sqrt{9-t^2}},1\right\rangle$

$\vec F(\vec r(t))=\langle -t,2\sqrt{9-t^2}\rangle$ $\longrightarrow$ $\vec F(\vec r(t))\cdot\vec r\,'(t)=\frac{+t^2}{\sqrt{9-t^2}}+2\sqrt{9-t^2}$

$$\int_{C_2}\vec F\cdot d\vec r=\int_{-\frac94}^0\frac{18-t^2}{\sqrt{9-t^2}}dt=\boxed{\frac{27\sqrt7}{32}+\frac{27}{2}\arcsin\left(\frac34\right)}$$

$=\frac{18-2t^2+t^2}{\sqrt{4-t^2}}=\frac{18-t^2}{\sqrt{9-t^2}}$

≈ 13.6812

y decreases

C_1: $y=t,\ x=\sqrt{-4t-t^2}$ $\quad$ $\vec r(t)=\langle\sqrt{-4t-t^2},t\rangle$ $\quad$ $t=0..-9/4$ $\quad$ $\vec r\,'(t)=\left\langle\frac{-2-t}{\sqrt{-4t-t^2}},1\right\rangle$

$\vec F(\vec r(t))=\langle -t,2\sqrt{-4t-t^2}\rangle$ $\longrightarrow$ $\vec F(\vec r(t))\cdot\vec r\,'(t)=\frac{t^2+2t}{\sqrt{-4t-t^2}}+2\sqrt{-4t-t^2}=\frac{-6t-t^2}{\sqrt{-4t-t^2}}$

$$\int_{C_1}\vec F\cdot d\vec r=\int_0^{-9/4}\frac{-6t-t^2}{\sqrt{-4t-t^2}}dt=\boxed{-6\pi+\frac{45\sqrt7}{32}+12\arctan\left(\frac{\sqrt7}{3}\right)}\approx -6.4562$$

$$\oint_C\vec F\cdot d\vec r=\int_{C_1}\vec F\cdot d\vec r+\int_{C_2}\vec F\cdot d\vec r+\int_{C_3}\vec F\cdot d\vec r=\boxed{\frac{9\sqrt7}{4}+\frac{27}{2}\arcsin\frac34-6\pi+12\underbrace{\arctan\frac{\sqrt7}{3}}}$$

$\frac{\pi}{2}-\underbrace{\arctan\frac{3}{\sqrt7}}_{\arcsin 3/4}$

$$=\boxed{\frac{9\sqrt7}{4}+\frac32\arcsin\left(\frac34\right) \qquad \approx 7.2250}$$

MAT2500-01/04 18S Final Exam Answers (2)

(1) line integrals in polar coordinates (alternate soln) : $\theta = t$

$\vec{r}(t) = \langle r(t)\cos(t),\ r(t)\sin t \rangle$

$C_1: r(t) = -4\sin t,\quad \vec{r}(t) = \langle -4\sin t\cos t, -4\sin^2 t\rangle,\ t = 0..+\Theta$

$\vec{r}\,'(t) = \langle -4\cos^2 t + 4\sin^2 t, -8\sin t\cos t\rangle,\quad \vec{F}(\vec{r}(t)) = \langle -4\sin^2 t, -8\sin t\cos t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 16(\cos^2 t - \sin^2 t)\sin^2 t + 64\sin^2 t\cos^2 t = 48\cos^2 t\sin^2 t - 16\sin^4 t$

Θ — arctan $3/\sqrt{7}$

$$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_0^{+\Theta} 48\cos^2 t\sin^2 t - 16\sin^4 t\,dt = \frac{45}{32}\sqrt{7} - \underbrace{12\arctan\left(\frac{3}{\sqrt{7}}\right)}_{+12\left(\arctan\frac{\sqrt{7}}{3} - \frac{\pi}{2}\right)}$$

$$= \frac{45}{32}\sqrt{7} - 6\pi + 12\arctan\frac{\sqrt{7}}{3}$$

$C_2: r(t) = 3,\ \vec{r}(t) = \langle 3\cos t, 3\sin t\rangle,\ t = \Theta..0$

$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t\rangle,\quad \vec{F}(\vec{r}(t)) = \langle -3\sin t, +6\cos t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 9\cos^2 t + 18\sin^2 t$

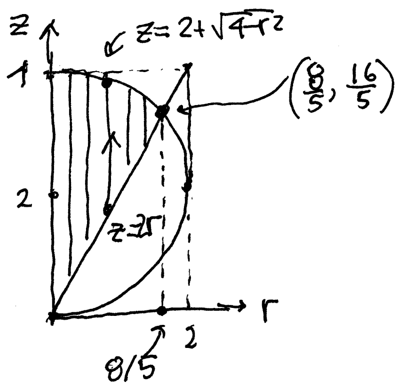
$$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{\Theta}^{0} 9\cos^2 t + 18\sin^2 t\,dt = \frac{27}{32}\sqrt{7} + \frac{27}{2}\arctan\left(\frac{3}{\sqrt{7}}\right)$$

$$\int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} = \frac{9}{4}\sqrt{7} + \frac{3}{2}\arctan\frac{3}{\sqrt{7}}$$

(2) a) $\underbrace{x^2+y^2}_{r^2} + (z-2)^2 = 4 \rightarrow \boxed{r^2 + (z-2)^2 = 4}$ $C(0,2)$, radius 2 $\rightarrow z - 2 = \pm\sqrt{4-r^2}$

$z = 2 \pm \sqrt{4-r^2}$

$z^2 = 4(x^2+y^2) = 4r^2 \rightarrow \boxed{z = 2r}$ $(z \geq 0)$ upper cone $\longrightarrow \boxed{z = 2 + \sqrt{4-r^2}}$



intersection:

$r^2 + z^2 \cancel{-4z} + \cancel{4} = \cancel{4}$ 0

$r^2 + z^2 = 4z$

$r^2 + (2r)^2 = 4(2r)$

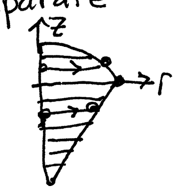
$5r^2 = 8r$

$(5r-8)r = 0$

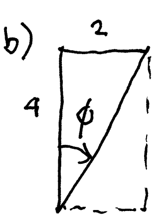
$r = 0,\ 8/5$

$z = 0,\ 16/5$

($r = r(z)$ for upper limit requires 2 separate integrals)

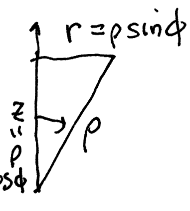


b)



$\tan\phi = \frac{2}{4} = \frac{1}{2}$

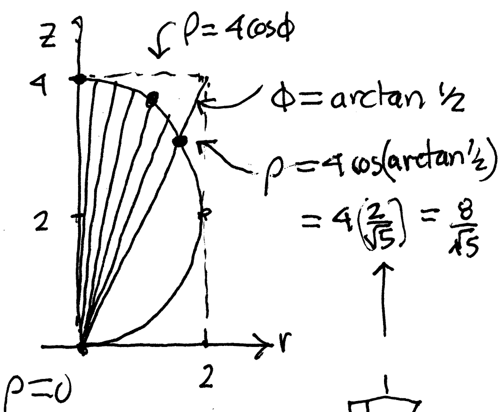
$\phi = \arctan 1/2$

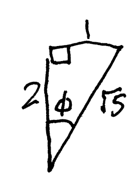


$r^2 + z^2 = 4z$

$\rho^2 = 4\rho\cos\phi$

$\begin{cases}\rho = 4\cos\phi\\ \rho = 0\end{cases}$





c) $\boxed{V = \int_0^{2\pi}\int_0^{8/5}\int_{2r}^{2+\sqrt{4-r^2}} (1)\,r\,dz\,dr\,d\theta = \frac{96}{25}\pi}$

d) $\boxed{V = \int_0^{2\pi}\int_0^{\arctan 1/2}\int_0^{4\cos\phi} (1)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{96}{25}\pi}$ ≈ 12.0637

MAT2500-01/04 18S Final Exam Answers (3)

3 a) $\vec{F} = \langle 1, \sin z, y\cos z \rangle$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(\sin z) + \frac{\partial}{\partial z}(y\cos z) = 0 + 0 + -y\sin z = -y\sin z$$

b) $$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \left\langle \underbrace{\frac{\partial}{\partial y}(y\cos z) - \frac{\partial}{\partial z}(\sin z)}_{1},\ \underbrace{\frac{\partial}{\partial z}(1) - \frac{\partial}{\partial x}(y\cos z)}_{2},\ \underbrace{\frac{\partial}{\partial x}\sin z - \frac{\partial}{\partial y}(1)}_{3} \right\rangle$$

$$= \langle \cos z - \cos z, 0, 0 \rangle = \langle 0, 0, 0 \rangle \checkmark$$

c) $\vec{\nabla} f = \vec{F}$:

$$\int \left[\frac{\partial f}{\partial x} = 1\right] dx \to f = \int 1\,dx = x + C(y,z) \to \frac{\partial f}{\partial y} = 0 + \frac{\partial C}{\partial y}(y,z)$$

$$\frac{\partial f}{\partial y} = \sin z = \frac{\partial C}{\partial y}(y,z) \to C(y,z) = \int \sin z\,dy = y\sin z + C(z)$$

$$\hookrightarrow f = x + y\sin z + C(z)$$

$$\frac{\partial f}{\partial z} = \cancel{y\cos z} = \cancel{y\cos z} + C'(z) \leftarrow \frac{\partial f}{\partial z} = 0 + y\cos z + C'(z)$$

$$C'(z) = 0, \quad C(z) = k = 0 \text{ (choice)}$$

$$\boxed{f = x + y\sin z}$$

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$ $t = 0..1 \to$ $\vec{r}(0) = \langle 0,0,0 \rangle$, $\vec{r}(1) = \langle 1,1,1 \rangle$

e) $$\int_C \vec{F}\cdot d\vec{r} = f(1,1,1) - f(0,0,0) = \boxed{1 + \sin 1} \approx 1.8415$$

d) $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{F}(\vec{r}(t)) = \langle 1, \sin t^3, t^2\cos t^3 \rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 1 + 2t\sin t^3 + 3t^4\cos t^3$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 1 + 2t\sin t^3 + 3t^4\cos t^3\,dt \overset{\text{Maple}}{=} \frac{(\tan\frac{1}{2} + 1)^2}{1 + \tan^2\frac{1}{2}} \overset{\text{simplify}}{=} \underbrace{2\sin\tfrac{1}{2}\cos\tfrac{1}{2}}_{= \sin 1} + 1 \approx 1.8415$$

sine double angle formula.