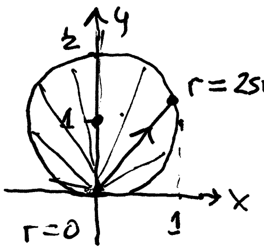
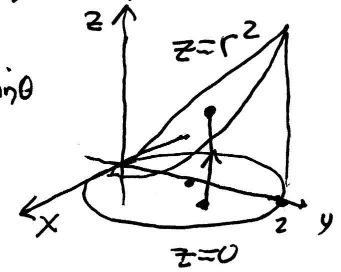
MAT2500-01/04 18S TakeHomeTest3 Answers (1)

① a) $x^2+y^2=2y,\quad x^2+y^2-2y=0$

$x^2+(y-1)^2-1=0,\quad x^2+(y-1)^2=1$

center: $(0,1)$ radius 1





$x^2+y^2=2y \rightarrow r^2=2r\sin\theta$

$\rightarrow r=2\sin\theta,\ \theta=0..\pi$

$$\boxed{\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} z\,\underbrace{r\,dz\,dr\,d\theta}_{dV}}$$

b)

$$=\frac{z^2 r}{2}\Big|_{z=0}^{z=r^2}=\frac{(r^2)^2}{2}r-0$$

$$=\frac{r^5}{2}$$

$$=\int_0^{\pi}\int_0^{2\sin\theta}\frac{r^5}{2}\,dr\,d\theta$$

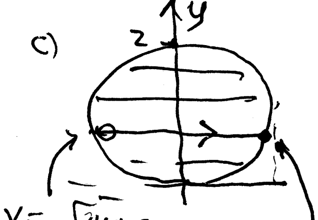
$$=\int_0^{\pi}\frac{1}{2}\frac{r^6}{6}\Big|_{r=0}^{r=2\sin\theta}d\theta=\int_0^{\pi}\frac{2^6}{12}\sin^6\theta\,d\theta$$

$$=\frac{16}{3}\int_0^{\pi}\sin^6\theta\,d\theta$$

$$=\frac{16}{3}\Big[\underbrace{-\frac{1}{6}\left(\sin^5\theta+\frac{5}{4}\sin^3\theta+\frac{15}{8}\sin\theta\right)\cos\theta}_{\rightarrow 0 \text{ at } \theta=0,\pi}+\frac{5\theta}{16}\Big]_0^{\pi}$$

$$=\frac{16}{3}\frac{5}{16}(\pi-0)=\boxed{\frac{5\pi}{3}}$$

c)

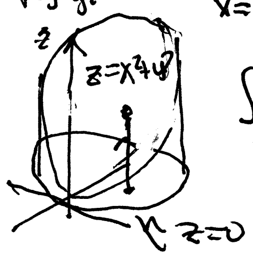


$x^2+y^2=2y$

$x^2=2y-y^2$

$x=\pm\sqrt{2y-y^2}$

$y=0..2$



$$\iiint_R z\,dV=\boxed{\int_0^2\int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}}\int_0^{x^2+y^2} z\,dz\,dx\,dy}$$

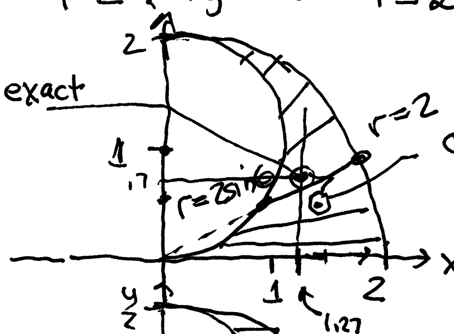
Maple $\boxed{\frac{5\pi}{3}}$ agrees!

② a) $x^2+y^2=2y \rightarrow x^2+(y-1)^2=1$ (left column)

$r^2=x^2+y^2=4\rightarrow r=2$

$\downarrow$

$r=2\sin\theta$ $\begin{bmatrix}x\ge 0, y\ge 0\\ \text{first quadrant}\end{bmatrix}$

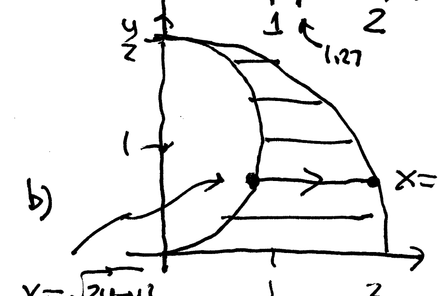


centroid guess?

$\theta=0..\pi/2$

seems "centered" in lower blob of region, below $y=1$ for sure, somewhere between $x=1$ & $x=2$

b)



$x=\sqrt{4-y^2}$

$y=0..2$

$$\langle A, A_y, A_x\rangle=\boxed{\int_0^2\int_{\sqrt{2y-y^2}}^{\sqrt{4-y^2}}\langle 1,x,y\rangle\,dx\,dy}$$

$$\overset{\text{Maple}}{=}\left\langle\frac{\pi}{2},2,\frac{8}{3}-\frac{\pi}{2}\right\rangle=A\langle 1,\bar{x},\bar{y}\rangle$$

$$\langle\bar{x},\bar{y}\rangle=\frac{\langle 2,\frac{8}{3}-\frac{\pi}{2}\rangle}{\frac{\pi}{2}}=\boxed{\left\langle\frac{4}{\pi},\frac{16}{3\pi}-1\right\rangle\approx\langle 1.27,0.70\rangle}$$

c) $$\langle A, A_y, A_x\rangle=\boxed{\int_0^{\frac{\pi}{2}}\int_{2\sin\theta}^{2}\langle 1,r\cos\theta,r\sin\theta\rangle\,r\,dr\,d\theta}$$

$\overset{\text{Maple}}{=}$ $\boxed{\text{same as above}}$

(d) exact versus guess? my hand diagram is not very good apparently, but even with Maple my guess would have been lower & to the right of the exact point! (see Maple plot.)

e) $\rho=\frac{k}{r}$: expect to push towards origin → more mass there

f) $$\langle M, M_y, M_x\rangle=\boxed{\int_0^{\frac{\pi}{2}}\int_{2\sin\theta}^{2}\langle 1,r\cos\theta,r\sin\theta\rangle\,\frac{k}{\cancel{r}}\,\cancel{r}\,dr\,d\theta}$$ cancel!

$\overset{\text{Maple}}{=}$ $k\langle\pi-2, 4/3, 2/3\rangle$

$$\langle\bar{x},\bar{y}\rangle=\frac{\langle\frac{4}{3},\frac{2}{3}\rangle}{\pi-2}=\boxed{\left\langle\frac{4}{3(\pi-2)},\frac{2}{3(\pi-2)}\right\rangle\approx\langle 1.17,0.58\rangle}$$

down to left compared to centroid.

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(3) $\underbrace{\underbrace{x^2+y^2}_{r^2}+z^2}_{\rho^2}=4$, $\underbrace{x^2+y^2}_{r^2}+(z-1)^2=1$, $z\ge 0$

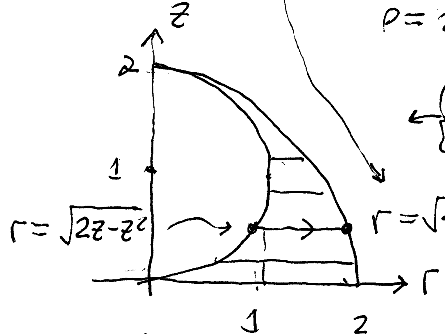
a) cyl: $r^2+z^2=4$, $r^2+(z-1)^2=1$, $z\ge 0$

sph: $\rho=2$, $r^2+z^2-2z+\cancel{1}=\cancel{1}$ → $r^2=2z-z^2$, $r=\sqrt{2z-z^2}$

$\rho^2-2\rho\cos\phi$

$\rho(\rho-2\cos\phi)=0$

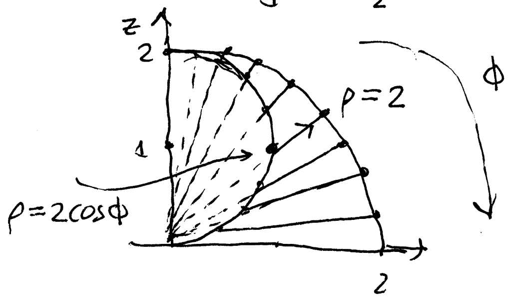
$\rho=2\cos\phi$



$(x,y)\to(r,z)$ problem 2 problem 3

$z=0..2$

only $dr\,dz$ order can be done with a single integral



$\phi=0..\pi/2$

$\theta=0..2\pi$ both coord systems

volume and "volume" moments:

b) $\langle V, V_{yz}, V_{zx}, V_{xy}\rangle$

$$\boxed{=\int_0^{2\pi}\int_0^2\int_{\sqrt{2z-z^2}}^{\sqrt{4-z^2}}\langle 1,\overbrace{r\cos\theta}^{x},\overbrace{r\sin\theta}^{y},z\rangle\, r\,dz\,dr\,d\theta}$$

$\downarrow$ 0 $\downarrow$ 0

$$\boxed{=\langle 4\pi,0,0,\tfrac{8\pi}{3}\rangle \approx \langle 12.57,0,0,8.37\rangle}$$

$\int_0^{2\pi}\langle\cos\theta,\sin\theta\rangle d\theta=0$

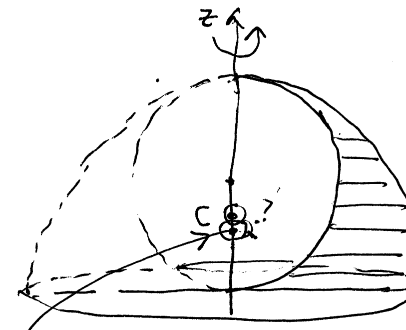
$\bar z=\dfrac{8\pi/3}{4\pi}=\boxed{\dfrac{2}{3}}$ $\boxed{\approx 0.67}$ seems right given the above diagram guess!

c) $\langle V, V_{yz}, V_{zx}, V_{xy}\rangle$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^{2}\langle 1,\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi\rangle\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$\downarrow$ 0 still zero $\downarrow$ 0

$=\boxed{\text{same as before}}$

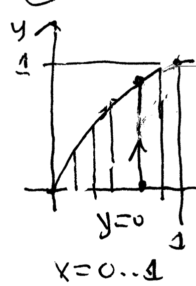


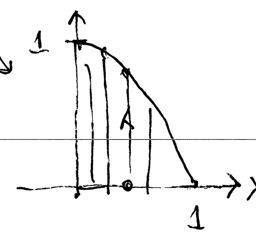
hemisphere with spherical cavity

centroid must lie on symmetry axis below halfway point (more volume below this).

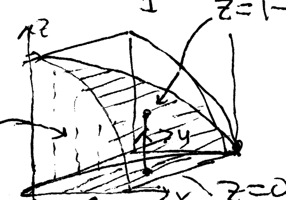
d) $z=2/3$ is close to my guess!

(4) a) $\int_{x=0}^{1}\int_{y=0}^{\sqrt{x}}\int_{z=0}^{1-x^2} f\,dz\,dy\,dx$





ceiling $z=1-x^2$



back wall, above xy parabola

z=0

floor

b) $x=y^2$ (back wall)

$x=\sqrt{1-z}$ ceiling

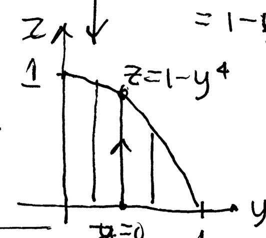
$z=1-x^2$
$x^2=1-z$
$x=\sqrt{1-z}$

project

intersection: $z=1-x^2$, $y=\sqrt{x}$ eliminate x to project onto yz plane

$z=1-x^2$, $x=y^2$

$=1-y^4$



$y=0..1$

$$\boxed{\int_0^1\int_0^{1-y^4}\int_{y^2}^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy}$$

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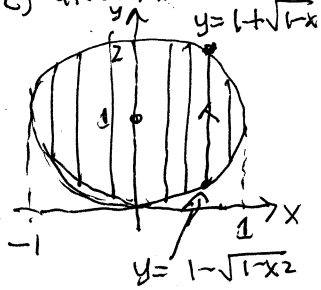
① c) alternative solution:



$$x^2+y^2-2y=0$$

$$\underbrace{y^2}_{a}\underbrace{-2y}_{b}+\underbrace{x^2}_{c}=0 \quad \text{quad form:}$$

$$y=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)x^2}}{2(1)}$$

$$=+1\pm\sqrt{1-x^2}$$

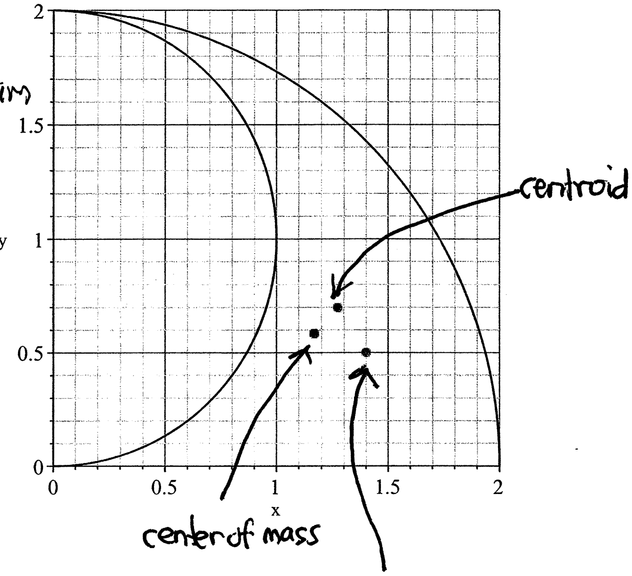
$x=-1..1$

$$\iiint_R z\,dV=\int_{-1}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}\int_0^{x^2+y^2} z\,dz\,dy\,dx$$

$$\overset{\text{maple}}{=}\frac{73}{48}\pi+\pi\infty \quad \text{nonsense}$$

Maple ran into some problem in this order of integration so when problems occur in one order of integration, try the opposite order. No problem arises then.

② diagram



bob guess for centroid
the tails near the vertical axis are a bit deceptive?

④ c) $\displaystyle\int_0^1\int_0^{\sqrt{x}}\underbrace{\int_0^{1-x^2} 1\,dz}\,dy\,dx$

$$z\Big|_{z=0}^{z=1-x^2}=1-x^2$$

$$\int_0^{\sqrt{x}}(1-x^2)\,dy=(1-x^2)y\Big|_{y=0}^{y=\sqrt{x}}$$

$$=(1-x^2)x^{1/2}=x^{1/2}-x^{5/2}$$

$$=\int_0^1 x^{1/2}-x^{5/2}\,dx=\frac{2x^{3/2}}{3}-\frac{2x^{7/2}}{7}\Big|_0^1$$

$$=\frac{2}{3}-\frac{2}{7}=2\left(\frac{7-3}{21}\right)=\boxed{\frac{8}{21}}$$

$$=\int_0^1\int_0^{1-y^4}\underbrace{\int_{y^2}^{\sqrt{1-z}} 1\,dx}\,dz\,dy$$

$$x\Big|_{x=y^2}^{x=\sqrt{1-z}}=\sqrt{1-z}-y^2$$

$$\int_0^{1-y^4}\sqrt{1-z}-y^2\,dz$$

u $du=-dz$ $\qquad -y^2z\Big|_{z=0}^{z=1-y^4}$

$$\int u^{1/2}(-du)=-\frac{2}{3}u^{3/2}=-\frac{2}{3}(1-z)^{3/2}$$

$$=-\frac{2}{3}(1-z)^{3/2}\Big|_{z=0}^{z=1-y^4}-y^2(1-y^4)$$

$$=-\frac{2}{3}(y^4)^{3/2}+\frac{2}{3}-y^2+y^6$$

$$=\int_0^1 -\frac{2}{3}y^6+\frac{2}{3}-y^2+y^6\,dy$$

$$=-\frac{2}{3}\frac{y^7}{7}+\frac{2}{3}y-\frac{y^3}{3}+\frac{y^7}{7}\Big|_0^1$$

$$=-\frac{2}{3\cdot 7}+\frac{1}{3}+\frac{1}{7}=-\frac{2}{21}+\frac{1}{3}+\frac{1}{7}=\cdots=\boxed{\frac{8}{21}}$$

maple (worn out) punt!

$$\left(=\frac{-2+7+3}{21}=\frac{8}{21}!\right)$$

yeah!

agree!