MAT2500-01/04 Test 2 Answers

① a) $f(x,y) = x^3 - 3xy + y^3$

$f_x(x,y) = \frac{\partial}{\partial x}(x^3 - 3xy + y^3) = 3x^2 - 3y + 0$
$= 3x^2 - 3y$

$f_{xx}(x,y) = \frac{\partial}{\partial x}(3x^2 - 3y) = 6x - 0 = 6x$

$f_y(x,y) = \frac{\partial}{\partial y}(x^3 - 3xy + y^3) = 0 - 3x + 3y^2$
$= -3x + 3y^2$

$f_{yx}(x,y) = \frac{\partial}{\partial x}(-3x + 3y^2) = -3 = f_{yx}(x,y)$

$f_{yy}(x,y) = \frac{\partial}{\partial y}(-3x + 3y^2) = 0 + 6y = 6y$

b) $f_x(x,y) = 0 = f_y(x,y)$

$3(x^2 - y) = 0 = 3(-x + y^2)$

$y = x^2$

$-x + (x^2)^2 = 0$

$x(-1 + x^3) = 0$

$x = 0,\ x = 1$

$y = 0,\ y = 1$

$\boxed{(0,0),\ (1,1) \text{ critical pts}}$ $\boxed{f(0,0) = 0,\ f(1,1) = -1}$

	$(0,0)$	$(1,1)$
f_{xx}	0	$6>0$ ⌣ local min
f_{yy}	0	$6>0$ ⌣ local min
f_{xy}	-3	-3
$f_{xx}f_{yy} - f_{xy}^2$	$-9<0$	$36-9>0$
	saddle	confirms 2D local min

c) $\vec{\nabla} f(2,0) = \langle f_x(2,0), f_y(2,0)\rangle$
$= \langle 3\cdot 2^2 - 0, -3(2) + 0\rangle = \langle 12, -6\rangle$
$= 6\langle 2, -1\rangle$

$\hat{u} = \widehat{\nabla f(2,0)} = \frac{\langle 2,-1\rangle}{\sqrt{4+1}} = \boxed{\frac{1}{\sqrt{5}}\langle 2,-1\rangle}$

$D_{\hat{u}} f(2,0) = |\vec{\nabla} f(2,0)| = \boxed{6\sqrt{5}}$

d) $\vec{v} = \langle 0,2\rangle - \langle 2,0\rangle = \langle -2,2\rangle = 2\langle -1,1\rangle$

$\hat{v} = \frac{\langle -1,1\rangle}{\sqrt{2}}$ $\quad D_{\hat{v}} f(2,0) = \hat{v}\cdot\vec{\nabla} f(2,0)$

$= \frac{1}{\sqrt{2}}\langle -1,1\rangle \cdot 6\langle 2,-1\rangle = \frac{6}{\sqrt{2}}(-2-1) = \boxed{-9\sqrt{2}}$

e) $F(x,y,z) = z - x^3 + 3xy - y^3$

$\vec{\nabla} F(x,y,z) = \langle -3x^2 + 3y, -3y^2 + 3x, 1\rangle$

$\vec{n} = \vec{\nabla} F(2,0,8) = \langle -3\cdot 2^2 + 0, 0 + 3(2), 1\rangle = \langle -12, 6, 1\rangle$

$\vec{r} = \langle 2,0,8\rangle + t\langle -12,6,1\rangle = \boxed{\langle 2-12t, 6t, 8+t\rangle = \langle x,y,z\rangle}$

$z = 0 = 8 + t \to t = -8$

$\vec{r} = \langle 2-12(-8), 6(-8), 0\rangle = \boxed{\langle 98, -48, 0\rangle}$

② $f(2,-1,1) = 2^2 + 2(-1)^2 + 3(1)^2 = 4 + 2 + 3 = 9$

a) $\boxed{x^2 + 2y^2 + 3z^2 = 9}$ level surface thru $(2,-1,1)$

b) $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $\vec{\nabla} f(x,y,z) = \langle 2x, 4y, 6z\rangle$

$\vec{\nabla} f(2,-1,1) = \langle 2(2), 4(-1), 6(1)\rangle = \langle 4,-4,6\rangle$
$= 2\langle 2,-2,3\rangle = \vec{n}$

$0 = \vec{n}\cdot\langle \vec{r} - \langle 2,-1,1\rangle\rangle = 2(x-2) - 2(y+1) + 3(z-1)$
$= 2x - 2y + 3z - 4 - 2 - 3 = 2x - 2y + 3z - 9$

$\boxed{2x - 2y + 3z = 9}$

c) $df = \vec{\nabla} f\cdot d\vec{r} = 2x\,dx + 4y\,dy + 6z\,dz$

$df(2,-1,1) = 4\,dx - 4\,dy + 6\,dz$

$d\vec{r} = \langle 1.95, -0.97, 1.02\rangle - \langle 2,-1,1\rangle$
$= \langle -0.05, 0.03, 0.02\rangle$

$df = 4(-0.05) - 4(0.03) + 6(0.02) = -0.20 - 0.12 + 0.12$
$= \boxed{-0.20 \approx \Delta f}$

$\frac{\Delta f}{f} \approx \frac{df}{f} = -\frac{0.20}{9} \approx -0.022 \sim \approx \boxed{2.2\% \text{ decrease}}$

③ $V = xyz$, $\frac{x}{3} + \frac{y}{2} + z = 1 \to z = 1 - \frac{x}{3} - \frac{y}{2} > 0$
$x > 0,\ y > 0$

$V = xy\left(1 - \frac{x}{3} - \frac{y}{2}\right) = xy - \frac{x^2y}{3} - \frac{xy^2}{2}$

$V_x = y - \frac{2xy}{3} - \frac{y^2}{2} = y\left(1 - \frac{2}{3}x - \frac{y}{2}\right) = 0$

$V_y = x - \frac{x^2}{3} - xy = x\left(1 - \frac{x}{3} - y\right) = 0$

$\frac{2x}{3} + \frac{y}{2} = 1 \xrightarrow{\times 2} \frac{4x}{3} + y = 2$

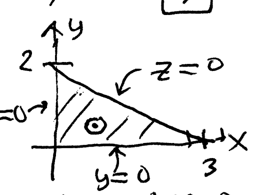
$\frac{x}{3} + y = 1 \xrightarrow{\times 1} \frac{x}{3} + y = 1$ $\quad y = 1 - \frac{x}{3} = 1 - \frac{1}{3} = \frac{2}{3}$

$-:\ x = 1$ $\quad z = 1 - \frac{1}{3}(1) - \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$

single crit pt: $\boxed{(x,y,z) = \left(1, \frac{2}{3}, \frac{1}{3}\right)}$

$V = xyz = 1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \boxed{\frac{2}{9}}$

optional explanation:



$V = 0$ on triangle boundary, $V > 0$ inside,
V continuous, only one critical pt → must be global max.

optional: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \to z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

$V = xy\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

↳ critical pt: $(x,y,z) = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

$V = \frac{abc}{27}$

obvious soln $(x,y,z) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ when $(a,b,c) = (1,1,1)$ by symmetry.
variables simply scale by these coefficients!