

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). Justify all claims.

1.  $x_1'(t) = 8x_1(t) + 11x_2(t) + 2x_3(t)$ ,  $x_2'(t) = -6x_1(t) - 9x_2(t) - 2x_3(t)$ ,  $x_3'(t) = 6x_1(t) + 6x_2(t) - x_3(t)$ ,  
 $x_1(0) = 5$ ,  $x_2(0) = -4$ ,  $x_3(0) = 5$ .

a) Rewrite this system of DEs and its initial conditions explicitly in matrix form for the vector variable  $\vec{x} = \langle x_1, x_2, x_3 \rangle$  as a column matrix (using the actual matrices, not their symbols), identifying the coefficient matrix  $A$  and suppressing function notation in the DE [ $x_1'$  not  $x_1'(t)$ ].

b) For this  $A$ , use Maple to write down the eigenvalues  $\lambda_1 \leq \lambda_2 \leq \lambda_3$  (ordered by increasing value, they are integers!) and corresponding matrix of eigenvectors  $B = \langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle$  that it provides you, reordering them if necessary to order them as requested.

c) Use technology to evaluate and write down the inverse matrix  $B^{-1}$  and use Maple to evaluate the matrix product  $A_B = B^{-1}AB$  (use periods for multiplication if necessary). Write down this result. Compare it to the corresponding list of eigenvalues in the same order as the eigenvectors.

d) Given that  $\vec{x} = B\vec{y}$ , if  $\vec{x}(0) = \langle 5, -4, 5 \rangle$ , use the inverse matrix to find  $\vec{y}(0)$  (integers!), showing the matrix multiplication intermediate step by hand (to prove that you actually know how to multiply a vector by a matrix, but check with Maple!).

e) Then write down and solve the decoupled equations for  $y_1, y_2, y_3$  and use their initial values to express the solution of the initial value problem for the original variables and finally write down the scalar solution  $x_1 = \dots$ , etc.

f) Enter this IVP from the first two lines above into Maple using "x1" etc for the subscripted variables to avoid wasting time with subscripts, but otherwise exactly as written. Right context menu it and solve the system. Does it agree with your hand solution? If not, write down the Maple solution and try to find your error.

① a) 
$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \underbrace{\begin{bmatrix} 8 & 11 & 2 \\ -6 & -9 & -2 \\ 6 & 6 & -1 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 5 \end{bmatrix}$$

e)  $\vec{y}' = A_B \vec{y} : \begin{matrix} y_1' = -3y_1 & y_1 = c_1 e^{-3t} & y_1(0) = c_1 = 1 \\ y_2' = -y_2 & y_2 = c_2 e^{-t} & y_2(0) = c_2 = 3 \\ y_3' = 2y_3 & y_3 = c_3 e^{2t} & y_3(0) = c_3 = 2 \end{matrix}$

b)  $\lambda = -3, -1, 2$   

$$B = \begin{bmatrix} -1 & 1 & 3/2 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ -2 & -2 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 3/2 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} \\ 3e^{-t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} -e^{-3t} + 3e^{-t} + 3e^{2t} \\ e^{3t} - 3e^{-t} - 2e^{2t} \\ 3e^{-t} + 2e^{2t} \end{bmatrix}$$

c)  $A_B = B^{-1}AB = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$  same order

$$\begin{bmatrix} x_1 = -e^{-3t} + 3e^{-t} + 3e^{2t} \\ x_2 = e^{3t} - 3e^{-t} - 2e^{2t} \\ x_3 = 3e^{-t} + 2e^{2t} \end{bmatrix}$$

f) Maple agrees

d)  $\vec{x} = B\vec{y} \rightarrow \vec{y} = B^{-1}\vec{x} \quad \vec{y}(0) = B^{-1}\vec{x}(0)$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -2 & -2 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0(5) + 1(-4) + 1(5) \\ -2(5) - 2(-4) + 1(5) \\ 2(5) + 2(-4) + 0(5) \end{bmatrix}$$

$$= \begin{bmatrix} 5-4 \\ -10+8+5 \\ 10-8 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$