MAT2705-04/05 18F Final Exam Answers (1)

① a) $\boxed{\begin{aligned} x_1 &= 2\cos 6t + 19\cos 7t - \cos 8t \\ x_2 &= \cos 6t + 3\cos 7t + 3\cos 8t \end{aligned}}$

b) $\omega_1 = 6 \qquad \omega_3 = 7 \qquad \omega_2 = 8$

$T_1 = \frac{2\pi}{6} = \frac{\pi}{3} \qquad T_3 = \frac{2\pi}{7} \qquad T_2 = \frac{2\pi}{8} = \frac{\pi}{4}$

since frequencies all integers, $\boxed{T = 2\pi}$ is a common period, but in this case there is no smaller common period.

$$6T_1 = 7T_3 = 8T_2 = 2\pi$$

periods of each mode in common period

c) $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -40 & 8 \\ 12 & -60 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -195\cos 7t \\ -195\cos 7t \end{bmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 20 \\ 7 \end{bmatrix}, \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

d) $A = \begin{bmatrix} -40 & 8 \\ 12 & -60 \end{bmatrix}$ maple → $\boxed{\begin{aligned} \lambda &= -36, -64 \\ B &= \begin{bmatrix} 2 & -1/3 \\ 1 & 1 \end{bmatrix} \end{aligned}}$ or exchanged

e) $0 = |A - \lambda I| = \begin{vmatrix} -40-\lambda & 8 \\ 12 & -60-\lambda \end{vmatrix}$

$= (\lambda+40)(\lambda+60) - 96$

$= \lambda^2 + 100\lambda + (2400-96) = x^2 + 100x + 2304$

Maple → $\lambda = \boxed{-36, -64 = \lambda_1, \lambda_2}$

$\lambda = -36$: $A + 36I = \begin{bmatrix} -40+36 & 8 \\ 12 & -60+36 \end{bmatrix}$

$= \begin{bmatrix} -4 & 8 \\ 12 & -24 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

L F

$x_2 = t,\ x_1 - 2x_2 = 0 \to x_1 = 2t$

$\langle x_1, x_2 \rangle = \langle 2t, t \rangle = t\underbrace{\langle 2, 1 \rangle}_{\vec{b}_1}$

$\lambda = -64$: $A + 64I = \begin{bmatrix} -40+64 & 8 \\ 12 & -60+64 \end{bmatrix}$

$= \begin{bmatrix} 24 & 8 \\ 12 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

L F

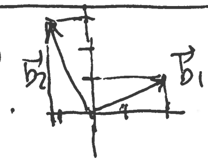
$x_2 = t,\ x_1 + \frac{1}{3}x_2 = 0 \to x_1 = -\frac{1}{3}t$

$\langle x_1, x_2 \rangle = \langle t, -t/3 \rangle = t\langle 1, -1/3 \rangle$

scale up ×3 → $\vec{b}_2 = \langle 3, -1 \rangle$

e) continued

$\boxed{B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix},\ B^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix},\ A_B = B^{-1}AB = \begin{bmatrix} -36 & 0 \\ 0 & -64 \end{bmatrix}}$

f) 

$\boxed{\begin{aligned} m_1 &= 1/2 \\ m_2 &= -3 \end{aligned}}$

need decimal values to compare with new coord axes — these look right ↓

g) $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 20 \\ 7 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3(20) + 1(7) \\ -1(20) + 2(7) \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 67 \\ -6 \end{bmatrix} \approx \begin{bmatrix} 9.6 \\ -0.86 \end{bmatrix}$

$B^{-1}\vec{F}(0) = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -195 \\ -195 \end{bmatrix} = -\frac{195}{7}\begin{bmatrix} 3+1 \\ -1+2 \end{bmatrix} = -\frac{195}{7}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -780/7 \\ -195/7 \end{bmatrix}$

$B^{-1}\begin{bmatrix} -7 \\ -7 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -7 \\ -7 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix} \qquad \vec{z} = -4\vec{b}_1 - \vec{b}_2$ like on graph

i) $\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -36 & 0 \\ 0 & -64 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 780/7 \\ 195/7 \end{bmatrix}\cos 7t$

$y_1'' + 36y_1 = -780/7 \cos 7t$

$y_2'' + 64y_2 = -195/7 \cos 7t$

$\boxed{\begin{aligned} y_{1h} &= c_1\cos 6t + c_2\sin 6t \\ y_{2h} &= c_3\cos 8t + c_4\sin 8t \end{aligned}}$

$y_{1p} = c_5\cos 7t + c_6\sin 7t$

$y_{1p}'' = -49(c_5\cos 7t + c_6\sin 7t)$

$y_{1p}'' + 36y_{1p} = \underbrace{(36-49)}_{-13}(c_5\cos 7t + \underbrace{c_6}_{c_6=0}\sin 7t) = -\frac{780}{7}\cos 7t$

$-13c_5 = -\frac{780}{7} \to c_5 = \frac{60}{7} \qquad \boxed{y_{1p} = \frac{60}{7}\cos 7t}$

$y_{2p} = c_7\cos 7t + c_8\sin 7t$

$y_{2p}'' = -49(c_7\cos 7t + c_8\sin 7t)$

$y_{2p}'' + 64y_{2p} = \underbrace{(64-49)}_{15}(c_7\cos 7t + \underbrace{c_8}_{c_8=0}\sin 7t) = -\frac{195}{7}\cos 7t$

$15c_7 = -\frac{195}{7} \to c_7 = -\frac{13}{7} \qquad \boxed{y_{2p} = -\frac{13}{7}\cos 7t}$

$\boxed{\begin{aligned} y_1 &= c_1\cos 6t + c_2\sin 6t + \tfrac{60}{7}\cos 7t \\ y_2 &= \underbrace{c_3\cos 8t + c_4\sin 8t}_{y_{ih}} \underbrace{- \tfrac{13}{7}\cos 7t}_{y_{ip}} \end{aligned}}$

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j) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \cos 6t + c_2 \sin 6t + 60/7 \cos 7t \\ c_3 \cos 8t + c_4 \sin 8t - 13/7 \cos 7t \end{bmatrix}$

done before ↓

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 + 60/7 \\ c_3 + 13/7 \end{bmatrix} = \begin{bmatrix} 20 \\ 7 \end{bmatrix}$ $\quad \begin{bmatrix} c_1 + 60/7 \\ c_3 - 13/7 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 20 \\ 7 \end{bmatrix} = \cdots = \begin{bmatrix} 67/7 \\ -6/7 \end{bmatrix}$

$c_1 = 67/7 - 60/7 = 1 \rightarrow y_{1h} = \cos 6t$

$c_3 = -6/7 + 13/7 = 1 \rightarrow y_{2h} = \cos 8t$

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -6c_1 \sin 6t - 6c_2 \cos 6t - 60 \sin 7t \\ -8c_3 \sin 8t - 8c_4 \cos 8t + 13 \sin 7t \end{bmatrix}$

$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -6c_2 \\ -8c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -6c_2 \\ -8c_4 \end{bmatrix} = B^{-1} \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c_2 = 0 \\ c_4 = 0 \end{matrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \cos 6t + 60/7 \cos 7t \\ \cos 8t - 13/7 \cos 7t \end{bmatrix} = \begin{bmatrix} 2\cos 6t - \cos 8t \\ \cos 6t + 3\cos 8t \end{bmatrix} + \underbrace{\begin{bmatrix} 2(60/7) - 1(-13/7) \\ 1(60/7) + 3(-13/7) \end{bmatrix}}_{\begin{bmatrix} 19 \\ 3 \end{bmatrix}} \cos 7t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \underbrace{2\cos 6t - \cos 8t}_{\vec{x}_h} + \underbrace{19 \cos 7t}_{\vec{x}_p} \\ \cos 6t + 3\cos 8t + 3 \cos 7t \end{bmatrix}$

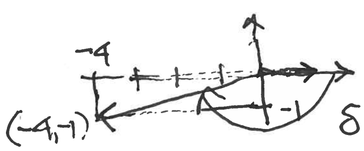
k) $= \underbrace{\overbrace{\cos 6t}^{y_{1h}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\text{tandem}} + \underbrace{\overbrace{\cos 8t}^{y_{2h}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{\text{accordian}} + \cos 7t \underbrace{\begin{bmatrix} 19 \\ 3 \end{bmatrix}}_{\equiv \vec{b}_3}$

tandem (same sign components)

l) $x_1 = 2\cos 6t - \frac{2}{3}\sin 6t \underbrace{- \cos 8t - \frac{1}{4}\sin 8t}$ $\qquad x_2 = \cos 6t - \frac{1}{3}\sin 6t + 3\cos 8t + \frac{3}{4}\sin 8t$

$x_{1(8)} = -1 \cos 8t - \frac{1}{4}\sin 8t$

$\hookrightarrow \langle -1, -\frac{1}{4} \rangle = \frac{\langle -4, -1 \rangle}{4}$



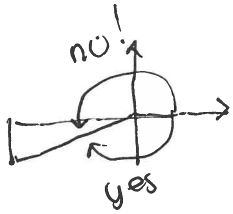
$\delta = -\pi + \arctan(1/4) \longrightarrow \frac{\delta}{2\pi} \approx -0.46$ cycles

graph shifts left on time axis
peaks earlier in time
compared to unshifted cosine
nearly half a cycle, i.e.
almost 180° out of phase.

$A = \frac{1}{4}\sqrt{16+1} = \frac{\sqrt{17}}{4}$

$x_{(8)} = \boxed{\frac{\sqrt{17}}{4} \cos(8t + \pi - \arctan 1/4)}$

δ:

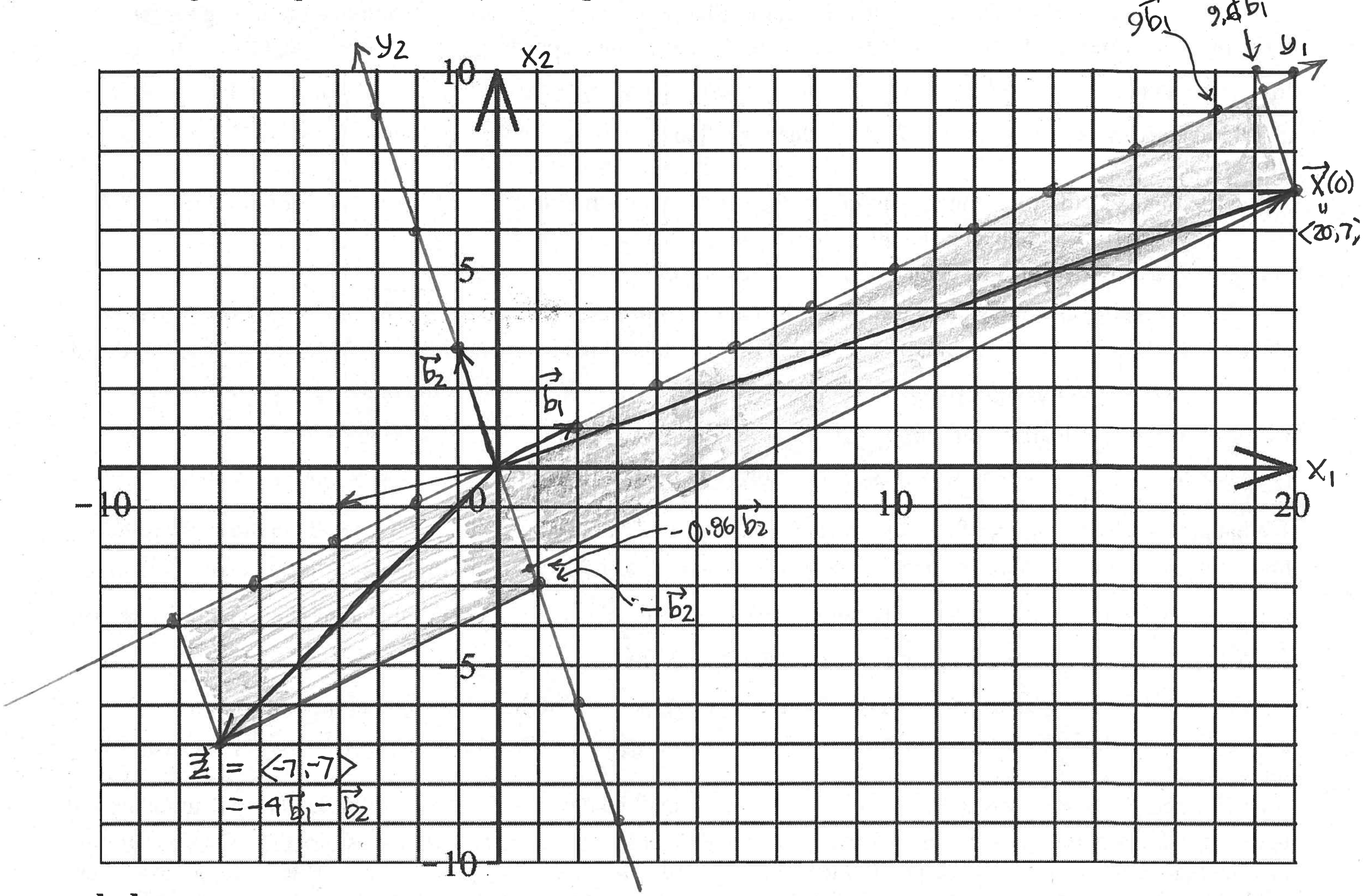


$|\delta| \le \pi \ (180°)!$

input the equations correctly?

k) Express the (correct) solution as a sum of the two eigenvector modes and the response mode in the form: $\vec{x} = y_{1h}\,\vec{b}_1 + y_{2h}\,\vec{b}_2 + \cos(7\,t)\,\vec{b}_3$ thus identifying the particular solution $\vec{x}_p$ (last term), the response vector coefficient $\vec{b}_3$ and the homogeneous solution $\vec{x}_h$ (first two terms), as well as the final expressions for the two decoupled variables y_{1h} and y_{2h}. Which homogeneous term is associated with the tandem mode and which with the accordian mode? Is the response term a tandem or accordian mode?

l) Use Maple to solve the undriven system $\vec{F} = \vec{0}$ with the initial conditions $x1(0) = 1, x2(0) = 4, x1'(0) = -6, x2'(0) = 4$. Write down the solution expression for x_1 and consider the term: $a\cos(\omega_2\,t) + b\sin(\omega_2\,t)$ in it. Plot the coefficient vector and evaluate its amplitude and phase shift exactly to re-express this function in phase-shifted cosine form. By what (numerical) fraction of a cycle does this sinusoidal function lag behind (peak later in time) or lead (peak before in time) the standard cosine function.



pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have not accessed any of the class web pages or any other sites during the exam. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: