MAT 2705-04/05 18F Takehome Test 3 Answers (1)

① a) $9x'' + 6x' + 10x = 0\,,\ x(0) = 1 = -x'(0)$

$x = e^{rt} \to 9r^2 + 6r + 10 = 0$

$r = \dfrac{-6 \pm \sqrt{36 - 4(9)10}}{2\cdot 9} = -\dfrac{6 \pm 6\sqrt{-9}}{3\cdot 6} = -\dfrac{1 \pm 3i}{3} = -\dfrac{1}{3} \pm i$

$e^{rt} = e^{-t/3} e^{\pm it} = e^{-t/3}(\cos t \pm i \sin t)$

$\hookrightarrow e^{-t/3}\cos t,\ e^{-t/3}\sin t$ (real basis)

$x = e^{-t/3}(c_1 \cos t + c_2 \sin t)$

$x' = -\frac{1}{3} e^{-t/3}(c_1 \cos t + c_2 \sin t) + e^{-t/3}(-c_1 \sin t + c_2 \cos t)$

$x(0) = c_1 = 1$

$x'(0) = -\frac{1}{3}c_1 + c_2 = -1 \to c_2 = -1 + \frac{1}{3}(1) = -2/3$

$\boxed{x = e^{-t/3}\left(\cos t - \frac{2}{3}\sin t\right)}$

$k_1 = \frac{1}{3} \to \boxed{\tau_1 = 3\,,\ \omega_1 = 1}$

$\frac{1}{9}(9x'' + 6x' + 10x) = 0$

$x'' + \frac{2}{3}x' + \frac{10}{9}x = 0$

$\boxed{k_0 = \frac{2}{3}}\quad \boxed{\omega_0 = \frac{\sqrt{10}}{3} \approx 1.054}$

$\boxed{\tau_0 = \frac{3}{2} = 1.5 \quad T_0 = \frac{6\pi}{\sqrt{10}} \approx 5.961}$

$\boxed{Q = \omega_0 \tau_0 = \frac{\sqrt{10}}{2} \approx 1.581}$

$\langle c_1, c_2 \rangle = \langle 1, -2/3 \rangle$

$A = \sqrt{c_1^2 + c_2^2} = \sqrt{1 + 4/9} = \frac{\sqrt{13}}{3}$

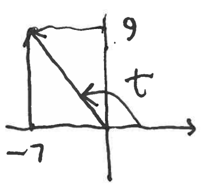
$\boxed{x = \pm \frac{\sqrt{13}}{3} e^{-t/3}}$ envelope functions

$x' = -\frac{1}{3} e^{-\frac{t}{3}}\left(\cos t - \frac{2}{3}\sin t\right) + e^{-t/3}\left(-\sin t - \frac{2}{3}\cos t\right)$

$= \frac{1}{3} e^{-t/3}\left((-1-2)\cos t + (\frac{2}{3} - 3)\sin t\right)$

$= \frac{1}{3} e^{-t/3}\underbrace{\left[-3\cos t - \frac{7}{3}\sin t\right]}_{} = 0$

$\tan t = -\frac{9}{7}$ need smallest $t > 0$ solution



$\boxed{t = \pi - \arctan 9/7 \approx 2.2318}$

$\boxed{x \approx -0.5419}$ (Maple)

b) $9x'' + 6x' + 10x = \underbrace{37 \sin \omega t}$

$\hookrightarrow x_p = c_3 \cos \omega t + c_4 \sin \omega t$

b) continued.

$10\,[x_p = c_3 \cos \omega t + c_4 \sin \omega t]$

$6\,[x_p' = -\omega c_3 \sin \omega t + \omega c_4 \cos \omega t]$

$9\,[x_p'' = -\omega^2 c_3 \cos \omega t - \omega^2 c_4 \sin \omega t]$

$9x_p'' + 6x_p' + 10x_p = [(10 - 9\omega^2)c_3 + 6\omega c_4]\cos \omega t$
$\quad + [-6\omega c_3 + (10 - 9\omega^2)c_4]\sin \omega t$
$\quad = 37 \sin \omega t$

$$\begin{bmatrix} 10 - 9\omega^2 & 6\omega \\ -6\omega & 10 - 9\omega^2 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 37 \end{bmatrix}$$

det: $\Delta = (10 - 9\omega^2)^2 + 36\omega^2 = 100 - 144\omega^2 + 81\omega^4$

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} 10 - 9\omega^2 & -6\omega \\ +6\omega & 10 - 9\omega^2 \end{bmatrix}\begin{bmatrix} 0 \\ 37 \end{bmatrix} = \frac{37}{\Delta}\begin{bmatrix} -6\omega \\ 10 - 9\omega^2 \end{bmatrix}$$

$A(\omega) = \sqrt{c_3^2 + c_4^2} = \frac{37}{\Delta}\left[(10 - 9\omega^2)^2 + 36\omega^2\right]^{1/2} = \frac{37}{\Delta^{1/2}}$

$$\boxed{A(\omega) = \frac{37}{\sqrt{81\omega^4 - 144\omega^2 + 100}}}$$

$0 = A'(\omega) = -\frac{1}{2}\Delta^{-3/2}(4\cdot 81\omega^3 - 2\cdot 144\omega)$

$= -18\Delta^{-3/2}\,\omega(9\omega^2 - 8)$

$\hookrightarrow \omega = 0,\ \frac{\sqrt{8}}{3} \to \boxed{\omega_{max} = \frac{2\sqrt{2}}{3} \approx 0.9428}$

$\boxed{A(\omega_{max}) \underset{\text{maple}}{=} \frac{37}{6} \approx 6.1667}$

$A(0) = \frac{37}{10}\,,\ \frac{A(\omega_{max})}{A(0)} = \frac{10}{6} = \boxed{\frac{5}{3} \approx 1.67}$

$Q \approx 1.58$ in same ball park

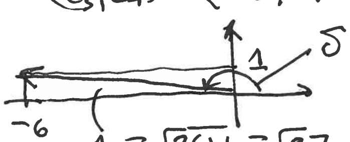
c) $\omega = 1$:

$\Delta = (10 - 9)^2 + 36 = 37$

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{37}{37}\begin{bmatrix} -6 \\ 10 - 9 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

$x_p = -6\cos t + \sin t$

$\langle c_3, c_4 \rangle = \langle -6, 1 \rangle$



$\tan \delta = -1/6$

$\boxed{\delta = \pi - \arctan 1/6}$

$A = \sqrt{36 + 1} = \sqrt{37}$

$\boxed{x_p = \sqrt{37}\cos(t - \pi + \arctan 1/6)}$

$\Delta\delta = \pi - \arctan 1/6 - \pi/2 = \frac{\pi}{2} - \arctan 1/6$

$\frac{\Delta\delta}{2\pi} \approx 0.224$ cycles > 0 so shifted right on time line, peaks later in time

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(2) a) $\boxed{x_1 = 5e^{-8t}\cos 2t\,,\quad x_2 = e^{-8t}(5\cos 2t - 10\sin t)}$

b) $x_1' = -9x_1 + x_2 \qquad x_1(0) = 5$

$x_2' = -5x_1 - 7x_2 \qquad x_2(0) = 5$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} -9 & 1 \\ -5 & -7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}}$$

c) $0 = |A - \lambda I| = \begin{vmatrix} -9-\lambda & 1 \\ -5 & -7-\lambda \end{vmatrix}$

$= (\lambda+9)(\lambda+7) + 5 = \lambda^2 + 16\lambda + 68$

$\xrightarrow[\text{Maple}]{} \lambda = -8 \pm 2i$

$\lambda = -8 + 2i$:

$A - \lambda I = \begin{bmatrix} -9+8-2i & 1 \\ -5 & -7+8-2i \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5}+\frac{2}{5}i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

L F

$x_2 = t$

$x_1 = (\frac{1}{5} - \frac{2}{5}i)\, x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5}(1-2i)t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} \frac{1}{5}(1-2i) \\ 1 \end{bmatrix}}_{\vec{b}_1} \to \vec{b}_2 = \overline{\vec{b}_1}$$

d)

$$\vec{x} = c_1 \underbrace{e^{(-8+2i)t}\,\vec{b}_1}\ + \text{c.c.}$$

$\to e^{-8t}(\cos 2t + i\sin 2t)\begin{bmatrix} \frac{1}{5}(1-2i) \\ 1 \end{bmatrix}$

$= e^{-8t}\begin{bmatrix} \frac{1}{5}\left((\cos 2t + 2\sin 2t) + i(\sin 2t - 2\cos 2t)\right) \\ \cos 2t + i\sin 2t \end{bmatrix}$

$= e^{-8t}\begin{bmatrix} \frac{1}{5}(\cos 2t + 2\sin 2t) \\ \cos 2t \end{bmatrix} + i\, e^{-8t}\begin{bmatrix} \frac{1}{5}(\sin 2t - 2\cos 2t) \\ \sin 2t \end{bmatrix}$ choose real basis of soln space

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a e^{-8t}\begin{bmatrix} \frac{1}{5}(\cos 2t + 2\sin 2t) \\ \cos 2t \end{bmatrix} + b e^{-8t}\begin{bmatrix} \frac{1}{5}(\sin 2t - 2\cos 2t) \\ \sin 2t \end{bmatrix}} \quad \text{(general soln)}$$

e) $\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = a\begin{bmatrix} 1/5 \\ 1 \end{bmatrix} + b\begin{bmatrix} -2/5 \\ 0 \end{bmatrix} = \begin{bmatrix} (a-2b)/5 \\ a \end{bmatrix} \to a = 5 \to 5 = \frac{5-2b}{5} \to b = -10$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-8t}\begin{bmatrix} \cos 2t + 2\sin 2t - 2\sin 2t + 4\cos 2t \\ 5\cos 2t - 10\sin t \end{bmatrix} = \boxed{\begin{bmatrix} e^{-8t}(5\cos 2t) \\ e^{-8t}(5\cos 2t - 10\sin 2t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ \text{(IVP soln)}}$$

f) $\tau = \frac{1}{8}$

$8\tau = 1$ so $t = 0..1$ in plot

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3) a)

$$x_1' = -x_1 \qquad x_1(0)=6$$
$$x_2' = x_1 - \tfrac{1}{2}x_2 \qquad x_2(0)=0$$
$$x_3' = \tfrac{1}{2}x_2 - \tfrac{1}{3}x_3 \qquad x_3(0)=0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1/2 & 0 \\ 0 & 1/2 & -1/3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

A triangular matrix, det equals product diagonal values!

b)

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 1 & -\frac{1}{2}-\lambda & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3}-\lambda \end{vmatrix} = -(\lambda+1)(\lambda+\tfrac{1}{2})(\lambda+\tfrac{1}{3}) = 0$$

$\boxed{\lambda = -1, -\tfrac{1}{2}, -\tfrac{1}{3}}$ ordered

c)

$\lambda = -\frac{1}{2}$:

$$A + \tfrac{1}{2}I = \begin{bmatrix} -1+\frac{1}{2} & 0 & 0 \\ 1 & -\frac{1}{2}+\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3}+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = 0 \\ x_2 + \frac{1}{3}t = 0 \to x_2 = -\frac{1}{3}t \end{array}$$

L L F

$x_3 = t$

$$\langle x_1, x_2, x_3 \rangle = \langle 0, -t/3, t \rangle = t \overbrace{\langle 0, -1/3, 1 \rangle}^{\vec{b}_2}$$

$\lambda = -\frac{1}{3}$:

$$A + \tfrac{1}{3}I = \begin{bmatrix} -1+1/3 & 0 & 0 \\ 1 & -\frac{1}{2}+\frac{1}{3} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -2/3 & 0 & 0 \\ 1 & -1/6 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$$

L L F

$x_3 = t$

$$\langle x_1, x_2, x_3 \rangle = \langle 0, 0, t \rangle = t \underbrace{\langle 0, 0, 1 \rangle}_{\vec{b}_3}$$

$\lambda = -1$:

$$A + I = \begin{bmatrix} -1+1 & 0 & 0 \\ 1 & -\frac{1}{2}+1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3}+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 4/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = 2/3\, t \\ x_2 = -4/3\, t \end{array}$$

L L F

$x_3 = t$

$$\langle x_1, x_2, x_3 \rangle = \langle 2/3\,t, -4/3\,t, t \rangle = t \underbrace{\langle 2/3, -4/3, 1 \rangle}_{\vec{b}_1}$$

c) continued

$\lambda = -1, -1/2, -1/3$ maple:

$$B = \begin{bmatrix} 2/3 & 0 & 0 \\ -4/3 & -1/3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 3/2 & 0 & 0 \\ -6 & -3 & 0 \\ 9/2 & 3 & 1 \end{bmatrix}$$

$$A_B = B^{-1}AB = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

d)

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -y_1 \\ -\frac{1}{2}y_2 \\ -\frac{1}{3}y_3 \end{bmatrix}$$

$y_1' = -y_1 \qquad y_1 = c_1 e^{-t}$

$y_2' = -\frac{1}{2}y_2 \qquad y_2 = c_2 e^{-t/2}$

$y_3' = -\frac{1}{3}y_3 \qquad y_3 = c_3 e^{-t/3}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & 0 \\ -4/3 & -1/3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-t/2} \\ c_3 e^{-t/3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}c_1 e^{-t} \\ -\frac{4}{3}c_1 e^{-t} - \frac{1}{3}c_2 e^{-t/2} \\ c_1 e^{-t} + c_2 e^{-t/2} + c_3 e^{-t/3} \end{bmatrix}}$$

general soln

e)

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = B^{-1} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 & 0 \\ -6 & -3 & 0 \\ 9/2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -36 \\ 27 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6e^{-t} \\ -12e^{-t} + 12e^{-t/2} \\ 9e^{-t} - 36e^{-t/2} + 27e^{-t/3} \end{bmatrix}}$$

(IVP soln)

f)

$$x_2' = 12(e^{-t} - \tfrac{1}{2}e^{-t/2}) = 0$$

$$2 = e^{t/2} \to t = 2\ln 2 \approx 1.3963$$

$$x_2 = 12(-e^{-2\ln 2} + e^{-\ln 2})$$
$$= 12(-(e^{+\ln 2})^{(-2)} + (e^{\ln 2})^{(-1)})$$
$$= 12(-\tfrac{1}{4} + \tfrac{1}{2}) = \boxed{3 = x_{2\,max}}$$

g) largest characteristic time $\tau_3 = 3 \to$ $5\tau_3 = 15$ for single exponential BUT in combination with others need a bit longer time window in plot.

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4) a) $x_1' = -x_2 \qquad x_1(0) = -2$

$x_2' = x_1 - \frac{5}{2}x_2 \qquad x_2(0) = 5$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & -5/2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}}$$

Maple:

$\lambda = -2, -1/2$

$B = \begin{bmatrix} 1/2 & 2 \\ 1 & 1 \end{bmatrix} \quad B^{-1} = -\frac{2}{3}\begin{bmatrix} 1 & -2 \\ -1 & 1/2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 & 4 \\ 2 & -1 \end{bmatrix}$

$A_B = B^{-1}AB = \begin{bmatrix} -2 & 0 \\ 0 & -1/2 \end{bmatrix}$

$\vec{x} = B\vec{y} \to \vec{y} = B^{-1}\vec{x} = \frac{1}{3}\begin{bmatrix} -2 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 4+20 \\ -4-5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 24 \\ -9 \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ -3 \end{bmatrix}} = \boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}$

b) $B^{-1}(B\vec{y})' = B^{-1}A(By) \to \vec{y}' = A_B\vec{y}$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2y_1 \\ -\frac{1}{2}y_2 \end{bmatrix} \qquad \begin{matrix} y_1' = -2y_1 \\ y_2' = -\frac{1}{2}y_2 \end{matrix} \qquad \begin{matrix} y_1 = c_1e^{-2t} \\ y_2 = c_2e^{-t/2} \end{matrix}$

$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$ from above

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 8e^{-2t} \\ -3e^{-t/2} \end{bmatrix} = \begin{bmatrix} 4e^{-2t} - 6e^{-t/2} \\ 8e^{-2t} - 3e^{-t/2} \end{bmatrix}$

c)

$k_1 = -2\ (2?) \qquad k_2 = -\frac{1}{2}\ (\frac{1}{2}?)$

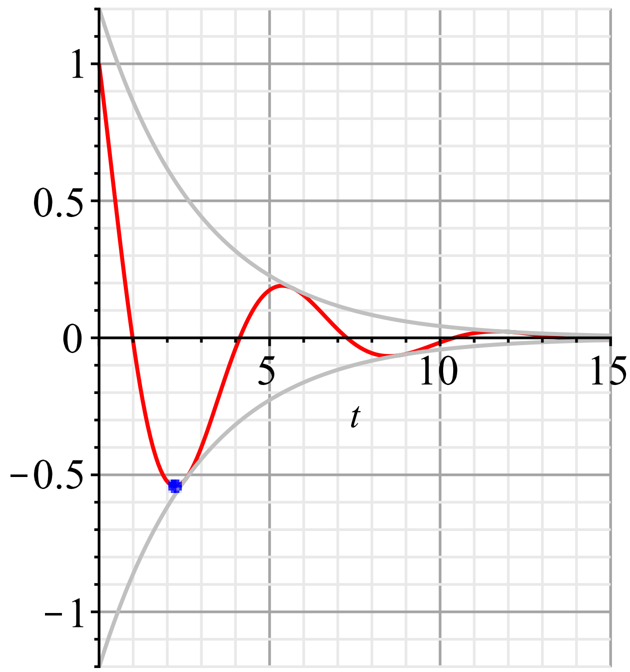
$\tau_1 = \frac{1}{2} \qquad \tau_2 = 2$

$5\tau_2 = 10$ initial trial & error window

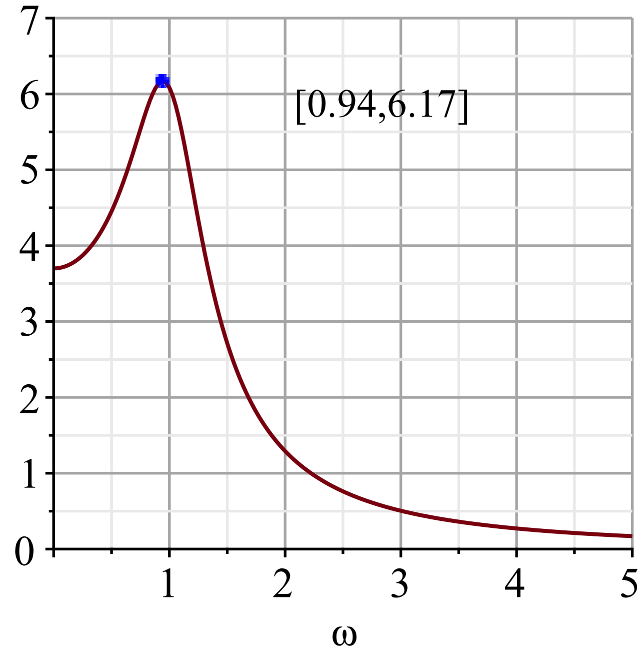
d) $x_1(10) = 4e^{-20} - 6e^{-5} \approx -6e^{-5} \approx \boxed{-0.0404}$ point lies on

$x_2(10) = 8e^{-20} - 3e^{-5} \approx -3e^{-5} \approx \boxed{-.0202}$ y_2 axis to this accuracy

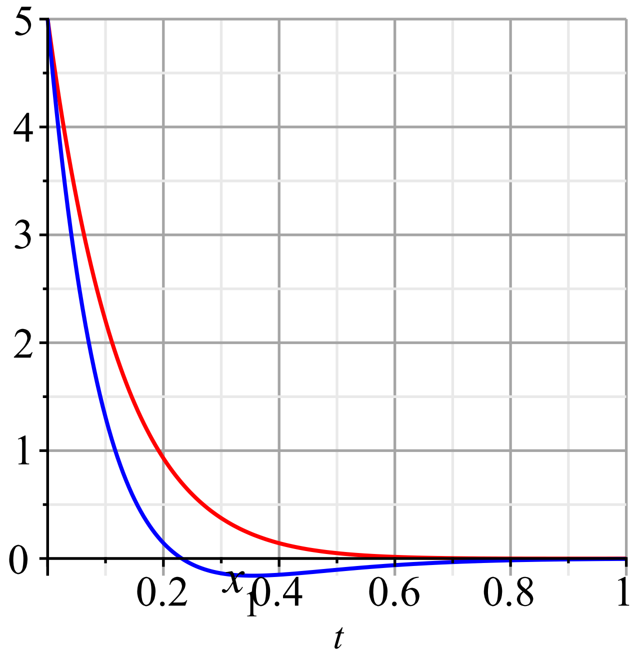
$\langle -0.0404, -0.0202 \rangle = -0.0202 \underbrace{\langle 2, 1 \rangle}_{\vec{b}_2}$



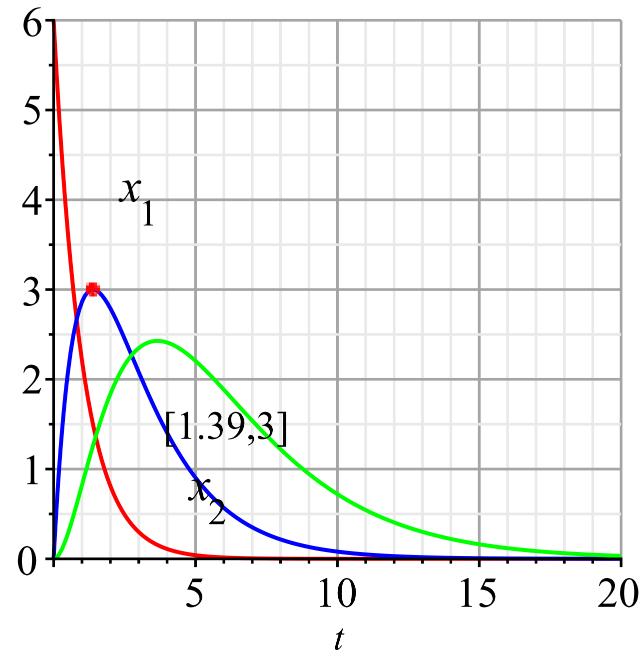
plot 1: response to extended impulse



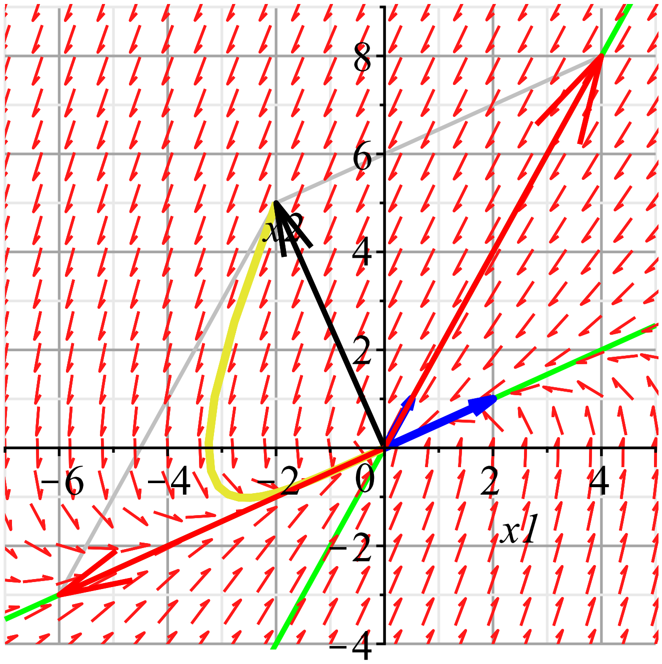
plot 2 : response amplitude $A(\omega)$



plot 3: x_1 (red) and x_2 (blue)



plot 4: 3 tank problem



plot 5: blue eigenvectors (label),
green coord axes (y_1, y_2 *upper*, *lower axes*),
black initial data vector (label),
red projections along axes,
yellow soln curve