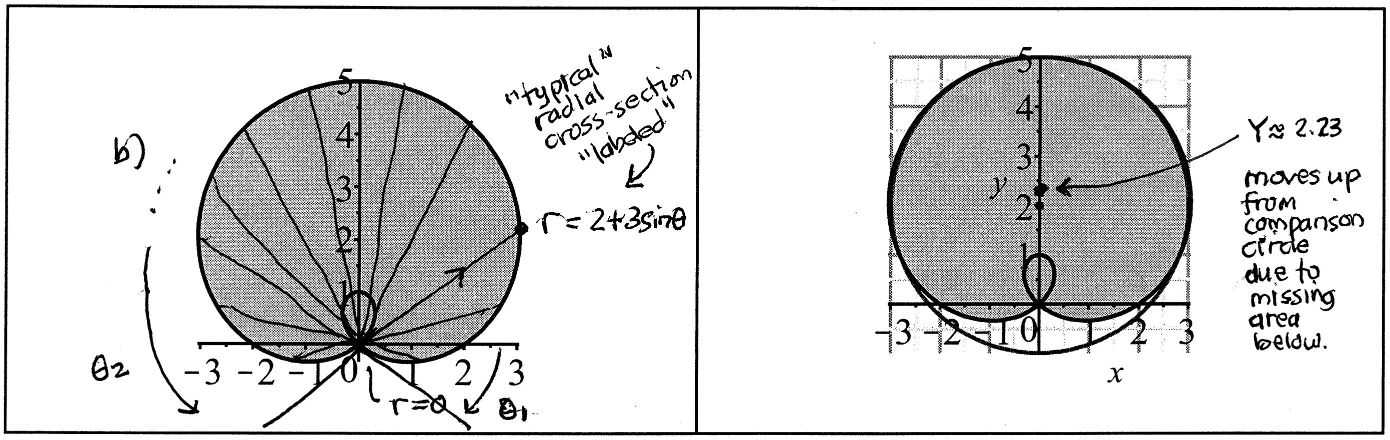
MAT2500-01/04 17S Quiz 8 Print Name (Last, First) ____________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).



Consider the region R of the plane inside the outer perimeter of the cardioid $r = 2+3\sin(\theta)$ shown in the figure. Find the position of its centroid on the y-axis for this region by following the steps:

a) First determine the negative acute angle $\theta_1 = -\theta_0$ (exactly using an inverse trig function, and to the nearest tenth of a degree) where the cardioid passes through the origin in the fourth quadrant, and passing along the cardioid in the counterclockwise direction, what is the positive polar angle θ_2 in the third quadrant at which the radius again returns to 0? So what is the angular interval of integration to describe this solid region of the plane where $r \ge 0$? Draw a reference triangle diagram illustrating the reference angle θ_0 to evaluate $\cos(\theta_1)$, $\sin(\theta_1)$ exactly from this diagram.

b) Draw in a typical radial cross-section with labeled bullet point endpoints (arrow in the middle etc) in the left figure together with a sampling of other such line segments to indicate the shaded region.

c) Set up the iterated double integral for the total area A and the two moments $A_y = \iint x\, dA$, $A_x = \iint y\, dA$ about the y and x axes respectively in polar coordinates.

d) Evaluate the three integrals exactly using Maple, and then the 4 decimal place value of the Y coordinate of the centroid. Indicate this point in the right figure.

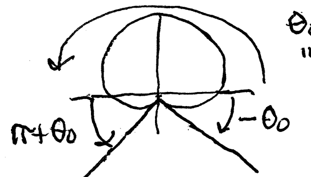
e) Evaluate numerically to 2 decimal places the fraction $\frac{A}{A_3}$ of the area compared to the area of the circle of radius 3 centered at $(0, 2)$ shown at the right. Does this correspond to what your eye sees?

solution

a) $r = 2+3\sin\theta = 0$

$\sin\theta = -2/3$

$\theta_1 = \boxed{\theta = -\arcsin 2/3 \approx -41.8°}$

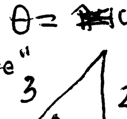


$\theta_0 = \arcsin 2/3$ "reference angle"

$\boxed{\theta_2 = \pi + \arcsin 2/3}$

so $\theta = -\arcsin 2/3 .. \pi + \arcsin 2/3$ (where $r \ge 0$)

"reference triangle"



$\sqrt{3^2-2^2} = \sqrt{5}$

$\begin{cases} \cos\theta_0 = \frac{\sqrt{5}}{3} \\ \sin\theta_0 = \frac{2}{3} \end{cases}$ $\boxed{\cos\theta_1 = \frac{\sqrt{5}}{3},\ \sin\theta_1 = -\frac{2}{3}}$

c) $\langle A, A_y, A_x \rangle = \int_{-\arcsin 2/3}^{\pi+\arcsin 2/3} \int_0^{2+3\sin\theta} \langle 1, \underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y} \rangle\, r\, dr\, d\theta$

d) $\overset{\text{Maple}}{=} \boxed{\left\langle \frac{17}{2}\arcsin\left(\frac{2}{3}\right) + 3\sqrt{5} + \frac{17}{4}\pi,\ 0,\ \frac{125}{18}\sqrt{5} + \frac{75}{4}\arcsin\left(\frac{2}{3}\right) + \frac{75}{8}\pi \right\rangle}$

$\approx \langle 26.3, 0, 58.7 \rangle$

$Y = \bar{y} = \frac{A_x}{A} \approx \boxed{2.2337}$

e) $A_3 = \pi \cdot 3^2 = 9\pi$

$\boxed{A/A_3 \approx 0.93}$

A 7% gap in the total area inside the (almost everywhere) outer circle looks right to me.

→ why needed:

$A = \int_{\theta_1}^{\theta_2} \int_0^{2+3\sin\theta} r\, dr = \int_{\theta_1}^{\theta_2} \frac{1}{2}(2+3\sin\theta)^2\, d\theta$

$= \cdots = \frac{17}{4}t - \frac{9}{4}\cos t \sin\theta - 6\cos t \Big|_{\theta_1}^{\theta_2}$

$\cos\theta_2 = -\cos\theta_1$, $\sin\theta_2 = \sin\theta_1$