MAT2500-01/04 17S Takehome Test 3 Answers (1)

① a) $\int_1^2\int_0^{8x-4x^2} x\,dy\,dx$

$= \int_1^2 \underbrace{xy\Big|_{y=0}^{y=8x-4x^2}}\,dy$

$x(8x-4x^2) = 8x^2-4x^3$

$= \int_1^2 8x^2-4x^3\,dx = 8\frac{x^3}{3} - 4\frac{x^4}{4}\Big|_1^2$

$= \frac{8}{3}(8-1)-(2^4-1) = \frac{56}{3}-15$

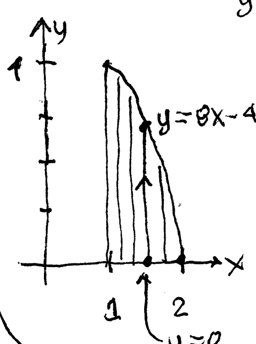
$= \frac{56-45}{3} = \boxed{\frac{11}{3}}$ Maple agrees.

b) $\int_{x=1}^{x=2}\int_{y=0}^{y=8x-4x^2} x\,dy\,dx$

$y = 4x(2-x) = 0$

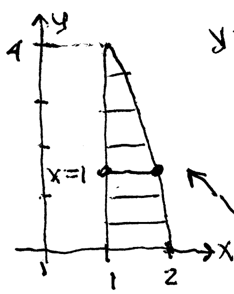
$\hookrightarrow x = 0, 2$ intercepts

vertex at $x=1, y=4$



$x = 1..2$

c)



$y = -4x^2+8x$

$4x^2-8x+y=0$

$x = \frac{8\pm\sqrt{64-4(4)(y)}}{2(4)}$

$= (1\pm\sqrt{1-y/4})$

$x = 1+\sqrt{1-y/4}$

$y = 0..4$

d) $\int_0^4\int_1^{1+\sqrt{1-y/4}} x\,dx\,dy$

e) $= \int_0^4 \frac{x^2}{2}\Big|_{x=1}^{x=1+\sqrt{1-y/4}} dy = \int_0^4 \frac{1}{2}\left(1+2\sqrt{1-y/4}+(1-y/4)-1\right)dx$

$= \int_0^4 \underbrace{\left(1-\frac{y}{4}\right)^{1/2}} + \frac{1}{2}-\frac{y}{8}\,dy = -\frac{8}{3}\left(1-\frac{y}{4}\right)^{3/2}+\frac{y}{2}-\frac{y^2}{16}\Big|_0^4$

$u = 1-\frac{y}{4}$

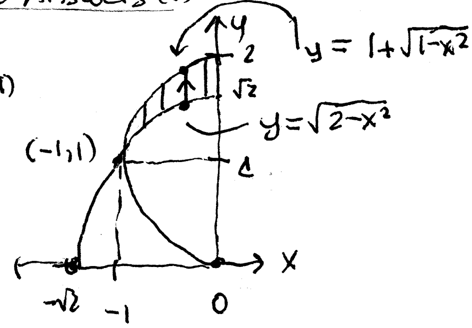
$du = -\frac{1}{4}dy$

$\int u^{1/2}(-4du) = -\frac{4u^{3/2}}{3/2}$

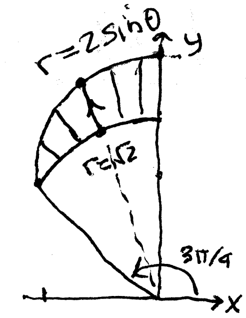
$= 0+2-\frac{16}{16}+\frac{8}{3} = 1+\frac{8}{3}$

$= \boxed{\frac{11}{3}}$ ✓

② a)



b)



$\theta = \frac{\pi}{2}..\frac{3\pi}{4}$

c) $\int_{\pi/2}^{3\pi/2}\int_{\sqrt{2}}^{2\sin\theta}(r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$

$= \int_{\pi/2}^{3\pi/4}\int_{\sqrt{2}}^{2\sin\theta} r^3\,dr\,\cos\theta\sin\theta\,d\theta$

$= \int_{\pi/2}^{3\pi/4}\frac{1}{4}\left(16\sin^4\theta-4\right)\cos\theta\sin\theta\,d\theta$

$= \int_{\pi/2}^{3\pi/4} 4\sin^5\theta-\sin\theta)\cos\theta\,d\theta$

$\left[\int 4u^5-u)\,du = 4\frac{u^6}{6}-\frac{u^2}{2}\right]$

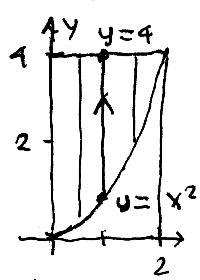
$= \frac{2}{3}\sin^6\theta-\frac{1}{2}\sin^2\theta\Big|_{\pi/2}^{3\pi/4}$

$= -\left(\frac{2}{3}-\frac{1}{2}\right)+\frac{2}{3}\frac{1}{2^{6/2}}-\frac{1}{2}\frac{1}{2^{2/2}}$

$= -\left(\frac{2}{3}-\frac{1}{2}\right)+\frac{1}{12}-\frac{1}{4} = -\frac{2}{3}+\left(\frac{1}{4}+\frac{1}{12}\right) = \boxed{-\frac{1}{3}}$

agrees with first form of integral!

③ a) $\underbrace{\int_{x=0}^{x=2}\int_{y=x^2}^{y=4}}\underbrace{\int_{z=0}^{z=\frac{y-x^2}{2}}} 1\,dz\,dy\,dx$ Maple: $\boxed{=\frac{64}{15}}$



$x = 0..2 \longrightarrow x = \pm\sqrt{y-2z}$

select positive root

$z = 0..\frac{y-x^2}{2}$

floor bottom $(z=0)$

ceiling top: $z = \frac{y-x^2}{2}$

or $y = 2z+x^2$

② a) $\int_{x=-1}^{x=0}\int_{y=\sqrt{2-x^2}}^{y=1+\sqrt{1-x^2}} xy\,dy\,dx$ $\left(\overset{\text{Maple}}{=} -\frac{1}{3}\right)$

$y = 1+\sqrt{1-x^2}$

$(y-1)^2 = 1-x^2$ $x^2+(y-1)^2=1$

$x^2+y^2-2y+1=1$ circle at $(0,1)$ radius 1

$x = -1..0$ $y = \sqrt{2-x^2}$

$y^2 = 2-x^2$

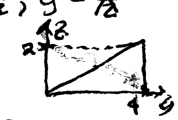
$x^2+y^2=2$ $\overset{\text{set}}{\Longleftrightarrow}$ $x^2+y^2=2y \longrightarrow y=1\to x=-1$ intersection pt

$r^2 = 2r\sin\theta$

b) $r^2=2$ $r=2\sin\theta$

$r=\sqrt{2}$ $=$ $\sin\theta = \frac{1}{\sqrt{2}}, \theta = -\arcsin\frac{1}{\sqrt{2}}+\pi = \frac{3\pi}{4}$

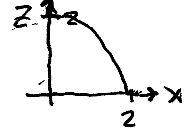
b) floor meets ceiling: $0 = y-\frac{x^2}{2}, y = \frac{x^2}{2}$



back meets ceiling $x=0$, $z = \frac{y-x^2}{2}$

$\hookrightarrow z = \frac{y}{2}, y = 2z$

side meets ceiling, $y=4$, $z = \frac{y-x^2}{2} = \frac{4-x^2}{2}$



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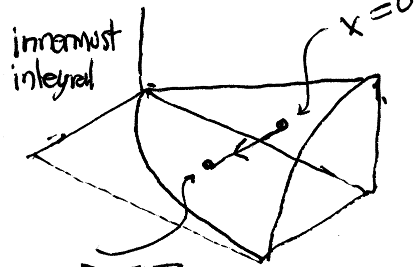
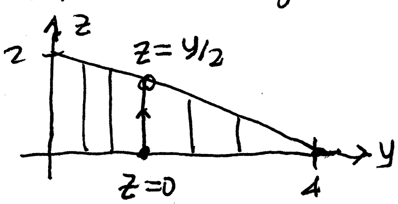
3 b) continued



$x=\sqrt{y-2z}$
ceiling for x

outer double integral:



$y = 0..4$

$$\boxed{\int_0^4\int_0^{y/2}\int_0^{\sqrt{y-2z}} 1\,dx\,dz\,dy \overset{\text{Maple}}{=} \frac{64}{15}}$$

agrees with original iteration value!

c) first octant: $\theta = 0..\pi/2$

$$V=\int_0^{\frac{\pi}{2}}\int_0^{\frac{8}{5}}\int_{2r}^{2+\sqrt{4-r^2}} 1\; r\,dz\,dr\,d\theta = \boxed{\frac{24}{25}\pi} \approx 3.02$$

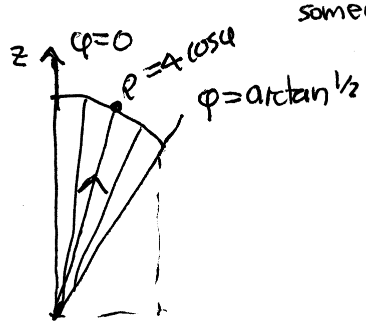
Maple

$$V_z=\int_0^{\frac{\pi}{2}}\int_0^{\frac{8}{5}}\int_{2r}^{2+\sqrt{4-r^2}} z\,r\,dz\,dr\,d\theta = \boxed{\frac{976}{375}\pi} \approx 8.18$$

$$\boxed{\bar{z} = \frac{V_z}{V} = \frac{122}{45} \approx 2.71}$$

it seems clear from the diagram it should be between 2 and 3 somewhere, so reasonable!

d)



$$V=\int_0^{\pi/2}\int_0^{\arctan 1/2}\int_0^{4\cos\varphi} 1\;\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$V_z=\int_0^{\pi/2}\int_0^{\arctan 1/2}\int_0^{4\cos\varphi} \underbrace{(\rho\cos\varphi)\,\rho^2\sin\varphi}_{\rho^3\sin\varphi\cos\varphi}\,d\rho\,d\varphi\,d\theta$$

4 a) $(z-2)^2+x^2+y^2=4$

sphere center $(0,0,2)$, radius 2

$z^2=4(x^2+y^2)$

cone: $z=\pm 2\sqrt{x^2+y^2} \to 2\sqrt{x^2+y^2}$ (first octant)

cylindrical: $\boxed{\begin{array}{l}(z-2)^2+r^2=4\\ z=2r\end{array}}$ or $\cancel{z^2-4z+4+r^2=4}$ not helpful yet.

$z-2=\pm\sqrt{4-r^2}$

$z=2\pm\sqrt{4-r^2}$

$z=2+\sqrt{4-r^2}$

sphere:

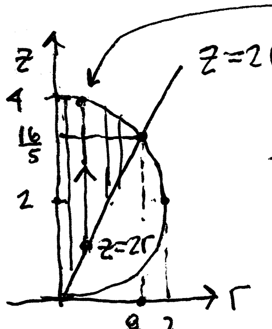
$\underbrace{r^2+z^2}=4z$

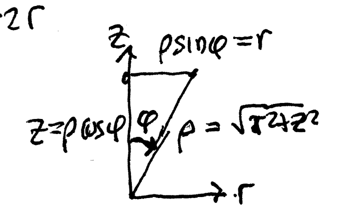
$\rho^2=4\rho\cos\varphi$

$\boxed{\rho=4\cos\varphi}$

spherical:

b)





cone:

$\rho\cos\varphi = 2(\rho\sin\varphi)$

$\tan\varphi = \frac{1}{2}$

$\boxed{\varphi = \arctan \tfrac{1}{2}}$

intersect:

$z=2r,\ r^2+z^2=4z$

$r^2+4r^2=8r$

$5r^2=8r$

$$\boxed{\begin{array}{l}r=8/5=1.6\\ z=16/5=3.2\end{array}}$$

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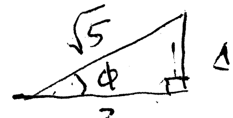
(3) $$\int_0^{\pi/2}\int_0^{\arctan 1/2}\int_0^{4\cos\varphi} 1\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} 1\, d\theta}_{\pi/2} \int_0^{\arctan 1/2} \underbrace{\int_0^{4\cos\varphi} \rho^2 d\rho}_{\frac{\rho^3}{3}\Big|_0^{4\cos\varphi} = \frac{64}{3}\cos^3\varphi} \sin\phi\, d\phi$$



$\tan\phi = 1/2$, $\cos\phi = \frac{2}{\sqrt{5}}$

$$\frac{64}{3}\int_0^{\arctan 1/2} \underbrace{\cos^3\varphi}_{u^3}\, \underbrace{\sin\phi\, d\phi}_{-du} = -\frac{64}{3\cdot 4}\cos^4\varphi\Big|_0^{\arctan 1/2} = \frac{16}{3}\Big(1-\underbrace{\cos^4(\arctan 1/2)}_{\frac{16}{25}}\Big)$$

$$-\frac{u^4}{4} = -\frac{\cos^4\varphi}{4}$$

$$= \frac{16}{3}\,\frac{9}{25} = \frac{48}{25}$$

$$= \frac{\pi}{2}\left(\frac{48}{25}\right) = \boxed{\frac{24}{25}\pi}$$ ✓

$$\int_0^{\pi/2}\int_0^{\arctan 1/2}\int_0^{4\cos\varphi} \rho^3 \sin\varphi\cos\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \underbrace{\int_0^{\pi/2} d\theta}_{\frac{\pi}{2}} \int_0^{\arctan 1/2} \underbrace{\int_0^{4\cos\varphi} \rho^3 d\rho}_{\frac{\rho^4}{4}\Big|_0^{4\cos\varphi} = 64\cos^4\phi} \sin\phi\cos\phi\, d\phi$$

$$\int_0^{\arctan 1/2} 64\cos^5\phi\, \sin\phi\, d\phi = -\frac{32}{3}\cos^6\phi\Big|_0^{\arctan 1/2}$$

$$\int u^5(-du) = -\frac{u^6}{6}$$

$$= \frac{32}{3}\Big(1-\underbrace{\cos^6(\arctan 1/2)}_{\frac{64}{125}}\Big)$$

$$= \frac{32\cdot 64}{3}\,\frac{(125-64)}{125} = \frac{1952}{375}$$

$$= \frac{\pi}{2}\left(\frac{1952}{375}\right) = \boxed{\frac{976}{375}\pi}$$ ✓

yeah!