MAT1505-03/04 Final Exam Answers (1)

1 a) $r = \dfrac{\sin\theta}{\theta}$,

$$r' = \frac{\theta\cdot\cos\theta - \sin\theta(1)}{\theta^2} = \frac{\theta\cos\theta - \sin\theta}{\theta^2}$$

$$r^2 + r'^2 = \frac{\sin^2\theta}{\theta^2} + \frac{\theta^2\cos^2\theta - 2\theta\cos\theta\sin\theta + \sin^2\theta}{\theta^4}$$

$$= \frac{\overbrace{\theta^2\sin^2\theta + \theta^2\cos^2\theta} + \sin^2\theta - 2\theta\cos\theta\sin\theta}{\theta^4}$$

$$= \frac{\theta^2 + \sin^2\theta - 2\theta\cos\theta\sin\theta}{\theta^4}$$

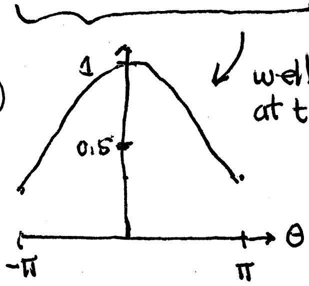
$$\sqrt{r^2+r'^2} = \sqrt{\frac{\theta^2+\sin^2\theta - 2\theta\cos\theta\sin\theta}{\theta^4}} = \sqrt{\frac{\text{numer}(\theta)}{\text{denom}(\theta)}}$$

$$= \frac{\sqrt{\theta^2+\sin^2\theta-2\theta\cos\theta\sin\theta}}{\theta^2}$$

$$L = \int_{-\pi}^{\pi}\sqrt{r^2+r'^2}\,d\theta = \int_{-\pi}^{\pi}\underbrace{\frac{\sqrt{\theta^2+\sin^2\theta-2\theta\cos\theta\sin\theta}}{\theta^2}}\,d\theta$$

$\underset{\text{Maple}}{\approx}$ 4.528346331

≈ 4.528 (4 digits)



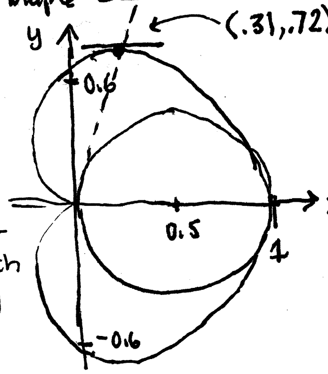
well-behaved at $t=0$

b) $A = \displaystyle\int_{-\pi}^{\pi}\frac{1}{2}r^2\,d\theta = \int_{-\pi}^{\pi}\frac{1}{2}\frac{\sin^2\theta}{\theta^2}\,d\theta$

$= \boxed{\mathrm{Si}(2\pi)} \approx 1.418151576$ $\boxed{\approx 1.418}$

Maple

c)



(.31,.72) looks right

poor sketch

$C_{circle} = 2\pi(\frac{1}{2}) = \pi \approx 3.14$

↳ $L \approx 4.53$ somewhat bigger as expected

$A_{circle} = \pi(\frac{1}{2})^2 = \frac{\pi}{4} \approx 0.79$

↳ $A \approx 1.418$ not quite twice as big — believable

d) $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{d\,(f(\theta)\sin\theta)/d\theta}{d\,(r(\theta)\cos\theta)/d\theta} = 0 \rightarrow$ numer $=0$

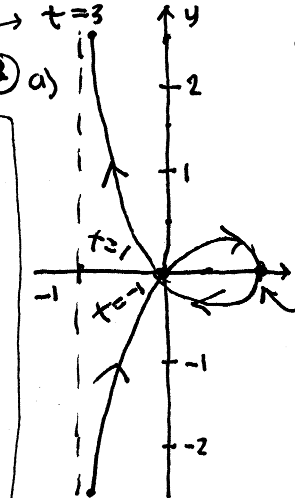
$$0 = \frac{d}{d\theta}\left(\frac{\sin\theta}{\theta}\sin\theta\right) = \frac{d}{d\theta}\left(\frac{\sin^2\theta}{\theta}\right) = \frac{\theta\cdot 2\sin\theta\cos\theta - \sin^2\theta(1)}{\theta^2}$$

$$= \frac{\sin\theta(2\theta\cos\theta - \sin\theta)}{\theta^2} \underset{\theta\neq 0}{\rightarrow} \boxed{\theta = \frac{1}{2}\frac{\sin\theta}{\cos\theta} = \frac{1}{2}\tan\theta} \text{ or } \boxed{2\theta\cos\theta = \sin\theta}$$

$\boxed{\theta \approx 1.165561185} \rightarrow x = \dfrac{\sin\theta}{\theta}\cos\theta \approx \boxed{0.3108}$, $y = \dfrac{\sin\theta}{\theta}\sin\theta \approx \boxed{0.7246}$

$\approx 67°$ for interpretation

2 a)



$t=-3$
$x = -4/5 = -0.8$
$y = -12/5 = -2.4$
reverse signs for endpoint

for $|t| \to \infty$
$x \sim -1$, $y \sim t$
so curve is traced out bottom to top.
$y = 0 \rightarrow t = 0, \pm 1$
$x = 1 \quad x = 0$

b) $\displaystyle\lim_{t\to\pm\infty} x = \lim_{t\to\pm\infty}\frac{1-t^2}{1+t^2} = \lim_{t\to\pm\infty}\frac{\frac{1}{t^2}-1}{\frac{1}{t^2}+1} = \boxed{-1}$

(vertical asymptote $x=-1$)

c) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \ldots$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right) = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$= \frac{-2t(+1+\cancel{t^2}+1-\cancel{t^2})}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{t^3-t}{1+t^2}\right) = \frac{(1+t^2)(3t^2-1) - (t^3-t)(2t)}{(1+t^2)^2}$$

$$= \frac{3t^2+3t^4-1-t^2-2t^4+2t^2}{(1+t^2)^2} = \frac{4t^2+t^4-1}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{4t^2+t^4-1}{-4t}$$

$$\left.\frac{dy}{dx}\right|_{t=\pm 1} = \mp\frac{4+1-1}{4} = \mp 1 \leftrightarrow \boxed{\pm 45° \text{ angle}}$$

$\boxed{\theta = \pm\pi/4}$

d) see next page

e) $A = 2\displaystyle\int_0^1 x\sqrt{\frac{1-x}{1+x}}\,dx \overset{\text{Maple}}{=} \boxed{2-\frac{\pi}{2}}$ $\approx \boxed{0.4292}$

f) $r = 2\cos\theta - \underbrace{\sec\theta}_{\frac{1}{\cos\theta}} = 0 \rightarrow 2\cos^2\theta - 1 = 0$

$\cos\theta = \pm 1/\sqrt{2}$

$\theta = \pm\pi/4$

optional

$r' = -2\sin\theta - \sec\theta\tan\theta$

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sqrt{r^2+r'^2}\,d\theta = 2\int_0^{\frac{\pi}{4}}\sqrt{r^2+r'^2}\,d\theta$$

$$r^2+r'^2 = \left(2c - \frac{1}{c}\right)^2 + \left(2s - \frac{s}{c^2}\right)^2$$

$$= 4c^2 - 4 + \frac{1}{c^2} + 4s^2 - \frac{4s^2}{c^2} + \frac{s^2}{c^4}$$

$$= \frac{c^2 - 4s^2c^2 + s^2}{c^4} = \frac{1-4s^2c^2}{c^4}$$

$$L = 2\int_0^{\frac{\pi}{4}}\frac{\sqrt{1-4\cos^2\theta\sin^2\theta}}{\cos^2\theta}\,d\theta \approx 2.490$$

② f) continued

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta = 2\int_0^{\frac{\pi}{4}} \frac{1}{2} (2\cos\theta - \sec\theta)^2 \, d\theta = \boxed{\int_0^{\frac{\pi}{4}} 4\cos^2\theta - 4 + \sec^2\theta \, d\theta}$$

$$\underset{\text{Maple}}{=} \boxed{2 - \frac{\pi}{2} \approx 0.4292} \text{ check } \checkmark$$

$-1 \le t \le 1$ for loop so

d) $A = \int_{-1}^{1} y(t) \, dx(t) = \int_{-1}^{1} \frac{t(t^2-1)}{1+t^2} \cdot \left(\frac{-4t}{(1+t^2)^2}\right) dt = \boxed{\int_{-1}^{1} \frac{4t^2(1-t^2)}{(1+t^2)^3} \, dt}$

$$\underset{\text{Maple}}{=} \boxed{2 - \frac{\pi}{2} \approx 0.4292} \text{ double check } \checkmark\checkmark \quad \left(\text{note: } = 2\int_0^1 \cdots dt\right)$$

①a) optional:

$\text{taylor}(\theta^2 + \sin^2\theta - 2\theta\cos\theta\sin\theta, \theta = 0, 7) = \theta^4 - \frac{2}{9}\theta^6 + O(\theta^8)$

$$\sqrt{\frac{\text{numer}(\theta)}{\text{denom}(\theta)}} = \sqrt{\frac{\theta^4(1 - \frac{2}{9}\theta^2 + \cdots)}{\theta^4}} = \sqrt{1 - \frac{2}{9}\theta^2 + \cdots}$$

so $\lim_{\theta\to 0} \sqrt{\frac{\text{numer}(\theta)}{\text{denom}(\theta)}} = \lim_{\theta\to 0} \sqrt{1 - \frac{2}{9}\theta^2 + \cdots} = 1$ well behaved:

② f) optional.

$$\frac{y^2}{x^2} = \frac{1-x}{1+x} \longrightarrow \frac{(t(t^2-1)/(1+t^2))^2}{(t(1-t^2)/(1+t^2))^2} \overset{?}{=} \frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}}$$

$$\frac{t^2}{t^2} \overset{?}{=} \frac{(1+t^2 - (1-t^2))/(1+t^2)}{(1+t^2 + (1-t^2))/(1+t^2)}$$

$$\overset{\checkmark}{=} \frac{2t^2}{2} = t^2 \quad \checkmark$$

polar form:

$$(\tan\theta)^2 = \frac{1 - r\cos\theta}{1 + r\cos\theta} \xrightarrow[r]{\text{solve for}} (1 + r\cos\theta)\tan^2\theta = 1 - r\cos\theta$$

$$\tan^2\theta + r\cos\theta\tan^2\theta =$$

$$r(\cos\theta\tan^2\theta + \cos\theta) = 1 - \tan^2\theta$$

$$r = \frac{1 - \tan^2\theta}{\cos\theta(\tan^2\theta + 1)} = \frac{1 - \tan^2\theta}{\cos\theta\sec^2\theta}$$

$$= \cos\theta(1 - \tan^2\theta) =$$

$$= \cos\theta(1 - (\sec^2\theta - 1))$$

$$= \cos\theta(2 - \sec^2\theta)$$

$$= 2\cos\theta - \sec\theta$$

$$\frac{\cos^2\theta + \sin^2\theta = 1}{\cos^2\theta \qquad \cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$

①a) optional:

$$\theta^2 + \sin^2\theta - 2\theta\cos\theta\sin\theta \underset{}{\overset{\text{Maple}}{=}} \theta^4 - \frac{2}{9}\theta^6 + O(\theta^8)$$

so

$$\frac{\theta^2 + \sin^2\theta - 2\theta\cos\theta\sin\theta}{\theta^4} = 1 - \frac{2}{9}\theta^2 + O(\theta^4)$$

$$\lim_{\theta\to 0} \frac{\theta^2 + \sin^2\theta - 2\theta\cos\theta\sin\theta}{\theta^4} = 1 \quad \checkmark$$