

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

1. a) Is the series  $\sum_{n=0}^{\infty} (-1)^n \frac{4}{(2n+1)2^{2n+1}}$  divergent, or absolutely or conditionally convergent? Explain.

b) What is the interval of convergence of the series  $\sum_{n=0}^{\infty} (-1)^n \frac{4 \cdot (2x-1)^{2n+1}}{(2n+1)2^{2n+1}}$ ? [Be sure to check the endpoints for convergence.]

c) Since this is an alternating series we can estimate the minimum number of terms we need to get the error to be less than  $10^{-5}$  for the series of part a). Do so. Is the actual error within this estimate? [Use Maple to get the numerical value of the exact infinite sum for part a).]

2. The ratio test is insensitive to factors  $f(n)$  whose "leading behavior" as the term number  $n$  goes to infinity is like some constant times  $n^p$  (in the sense of the limit comparison test:  $\lim_{n \rightarrow \infty} \frac{f(x)}{n^p} = \text{constant}$ ), which is why we have to resort to comparison with  $p$ -series when the ratio test fails. The same is true of factors whose leading behavior is like powers of  $\ln(x)$ , namely the ratio test is insensitive to them. The harmonic  $p = 1$  series is the minimum value of  $p$  where convergence is lost. Use the integral test to determine for what values of  $p$  additional

factors of  $\ln(n)$  make the harmonic series converge, i.e., for what values of  $p$  does  $\sum_{n=2}^{\infty} \frac{1}{n \cdot (\ln(n))^p}$  converge?

[We have to start at  $n = 2$  since division by zero occurs for  $n = 1$ .]

3. The period of oscillation of a pendulum of length  $L$  with maximum displacement angle  $\theta_0$  ( $g$  is the gravitational constant) is given by the formula

$$T = 4 \sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2(x)}} dx, \text{ where } k = \sin\left(\frac{\theta_0}{2}\right).$$

a) Use the binomial theorem to write out and simplify the first 3 terms of the integrand

$f(x) = \frac{1}{\sqrt{1 - k^2 \sin^2(x)}}$ , and then use Maple to evaluate the definite integral of those terms, and finally

express the approximate formula for the period using them, expressed in terms of powers of  $k$ , factoring out the leading term  $T_0 = 2\pi \sqrt{\frac{L}{g}}$  familiar (?) from high school or freshman physics.

b) For  $\theta_0 = \frac{\pi}{6}$ , where the half-angle formula leads to  $k = \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$ , evaluate each of the 3 terms of your approximation to  $\frac{T}{T_0}$  numerically and their sum.

c) Use Maple to evaluate the ratio  $\frac{T}{T_0}$  exactly for this angle and then numerically evaluate it to compare with your approximation to  $\frac{T}{T_0}$  of part b). Evaluate the ratio  $\frac{\text{approx} - \text{exact}}{\text{exact}}$ . What is the percentage error? What is the percentage error if only the first two terms are used? If the first term alone is used?

4. The inverse sine function has the Taylor series expansion  $\arcsin(x) = \sum_{n=0}^{\infty} \frac{(2n)! x^{2n+1}}{(n!)^2 4^n (2n+1)}$ . You can verify this with the Maple assumption that we stay within its interval of convergence:

> `assume( -1 < x < 1);`  $\sum_{n=0}^{\infty} \frac{(2n)! x^{2n+1}}{(n!)^2 4^n (2n+1)}$ ; `simplify(% , symbolic)`

a) Confirm that the radius of convergence is 1 using the absolute convergence ratio test.

b) Use the Stirling approximation  $n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$  for large  $n$  to test this series for convergence at the right endpoint  $x = 1$ . Simply replace the factorials by this formula and simplify and compare to a  $p$ -series for large  $n$  to accomplish this goal. [What does Maple know about the value of this series at  $x = 1$ ?]

c) The identity  $\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$  can be used to derive the above infinite series by integrating the

binomial expansion of the integrand term by term. Confirm that the first 3 terms of this expression coincide with those of the above formula for the infinite series.

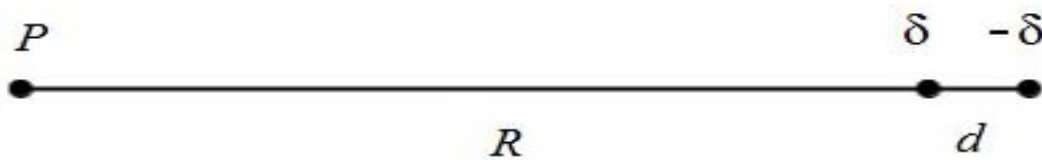
d) **Optional.** Show how one can actually find that formula by noticing that, for example, in the  $n = 3$  term:

$$1 \cdot 3 \cdot 5 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{2 \cdot 4 \cdot 6} = \frac{6!}{2^3 3!} = \frac{(2n)!}{2^n n!}$$

e) Numerically evaluate the first 8 terms in the series expansion for  $\arcsin(0.25)$ . Looking at these values, what is the minimum number of terms one *seems* to need to get 10 digit accuracy? Add only this minimum number of terms and compare to the direct 10 digit numerical approximation of  $\arcsin(0.25)$ . How do they compare? [Note that this angle is about  $14.3^\circ$ .]

5. In the same way that the inverse square electric fields of a pair of equal but oppositely signed nearby charges cancel out at large distances leaving an inverse cubic "dipole field" as the residual electric field, the inverse cube dipole fields of two equal but oppositely signed dipoles lead to an inverse quartic field at large distances.

Evaluate the lowest order Taylor approximation to its field  $E = \frac{\delta}{R^3} - \frac{\delta}{(R+d)^3}$  using the binomial series expansion.



It might be useful to remember how to make a sequence of numerical values (Maple function  $f$ ):

> `seq( evalf( f(n) ), n = 0 .. 10)`

## ► solution

## ▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet in on top of your answer sheets as a cover page, with the first test page facing up:

"During this examination, all work has been my own. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature:

Date: