# MAT1505-03/04 17F Test 2 Answers

① $y=\cos x$, $\frac{dy}{dx}=\sin x$,

$$L=\int_{-\pi/2}^{\pi/2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\boxed{\int_{-\pi/2}^{\pi/2}\sqrt{1+\sin^2 x}\,dx}$$

$$\overset{\text{Maple}}{=}\boxed{2\sqrt{2}\,\text{EllipticE}\left(\frac{\sqrt{2}}{2}\right)\approx 3.820}$$

② $$\int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}(1+x)}\,dx$$

integrand → ∞ at $x=0$

$$\overset{\text{Maple}}{=}\lim_{a\to 0^+} 2\arctan(\sqrt{x})\Big|_a^1 =$$

$$=\lim_{a\to 0^+}(2\arctan 1-\underbrace{2\arctan\sqrt{a}}_{\to 0})$$

$$=2\arctan(1)=2\left(\frac{\pi}{4}\right)=\boxed{\frac{\pi}{2}\approx 1.57080}$$

↖ 1.570796

③ a) $$\int_0^\infty p(x)\,dx=\int_0^\infty \frac{6x}{(x+1)^4}\,dx$$

$$=\lim_{a\to\infty}\int_0^a \frac{6x}{(x+1)^4}\,dx \overset{\text{Maple}}{=} \lim_{a\to\infty} \frac{2}{(1+x)^3}-\frac{3}{(1+x)^2}\Big|_0^a$$

$$=\lim_{a\to\infty}\underbrace{\frac{2}{(1+a)^3}}_{\to 0}-\underbrace{\frac{3}{(1+a)^2}}_{\to 0}-\underbrace{(2-3)}_{-1}=1\ \checkmark$$

b) $$\mu=\int_0^\infty x\,p(x)\,dx=\int_0^\infty \frac{6x^2}{(x+1)^4}\,dx$$

$$=\lim_{a\to\infty}\int_0^a \frac{6x^2\,dx}{(x+1)^4}$$

$$\overset{\text{Maple}}{=}\lim_{a\to\infty}\left(-\frac{2}{(1+x)^3}+\frac{6}{(1+x)^2}-\frac{6}{1+x}\right)\Big|_0^a$$

$$=\lim_{a\to\infty}\underbrace{\frac{-2}{(1+a)^3}}_{\to 0}+\underbrace{\frac{6}{(1+a)^2}}_{\to 0}-\underbrace{\frac{6}{1+a}}_{\to 0}-(-2+6-6)$$

$$=\boxed{2}$$

c) $$P(0\le x\le 2)=\int_0^2 p(x)\,dx=\int_0^2\frac{6x}{(x+1)^4}\,dx$$

$$=\frac{2}{(1+x)^3}-\frac{3}{(1+x)^2}\Big|_0^2=\frac{2}{3^3}-\frac{3}{3^2}-(2-3)$$

$$=1+\underbrace{\frac{2}{27}-\frac{1}{3}}_{-\frac{7}{27}}=\boxed{\frac{20}{27}\approx 0.741}$$

④ a) $$E_{avg}=\frac{2\pi N}{(\pi kT)^{3/2}}\int_0^\infty E^{3/2}e^{-E/kT}\,dE$$

$x=\frac{E}{kT}$, $dx=\frac{dE}{kT}$ $\quad \cancel{x}E=0\to x=0$

$\qquad\qquad\qquad\qquad\qquad E\to\infty\to x\to\infty$

$E=kTx$, $dE=kT\,dx$

$$E_{avg}=\frac{2\pi N}{(\pi kT)^{3/2}}\int_0^\infty (xkT)^{3/2}e^{-x}(kT\,dx)$$

$$=\frac{2\pi N}{\pi^{3/2}}\frac{\cancel{(kT)^{3/2}}\cancel{(kT)}}{\cancel{(kT)^{3/2}}}\int_0^\infty x^{3/2}e^{-x}\,dx$$

$$=NkT\boxed{\underbrace{\left(\frac{2}{\pi^{1/2}}\int_0^\infty x^{3/2}e^{-x}\,dx\right)}_{C}}$$

$$C\overset{\text{Maple}}{=}\frac{2}{\cancel{\pi^{1/2}}}\left(\frac{3}{4}\cancel{\pi^{1/2}}\right)$$

b)

$$=\boxed{\frac{3}{2}}$$

so $$\boxed{E_{avg}=\frac{3}{2}NkT}$$

(final formula)