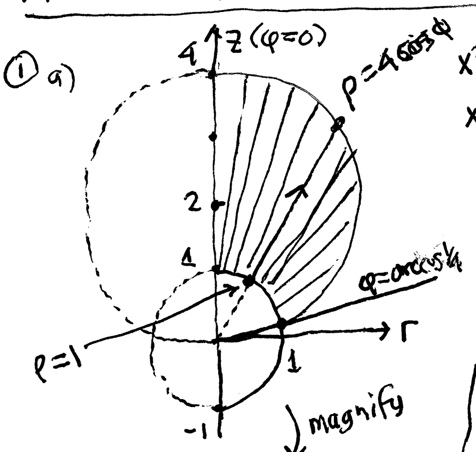
MAT2500-01/04 16S Final Exam Answers

① a)



$x^2+y^2+z^2=1 \rightarrow \rho=1$

$x^2+y^2+(z-2)^2=4$

$\downarrow$

$r^2+z^2=1$

$r^2+(z-2)^2=4$

$r^2+z^2-4z+4=4$

$r^2+z^2=4z$

$4z=1$

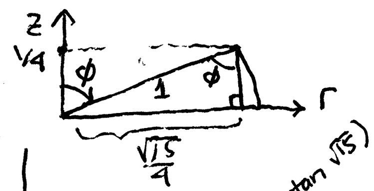
$\boxed{z=1/4}$

$r^2+\left(\frac{1}{4}\right)^2=1$

$r^2=1-\frac{1}{16}=\frac{15}{16}$

$\boxed{r=\frac{\sqrt{15}}{4}}\ \boxed{\approx 0.97}$

$\downarrow$ magnify

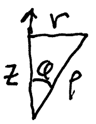


$\cos\phi=\frac{1/4}{1}=\frac{1}{4}$ ($=\arctan\sqrt{15}$)

$\boxed{\begin{aligned}\phi&=\arccos 1/4\\ &\approx 75.5^\circ\end{aligned}}$

$\rho^2=4(\rho\cos\phi)$

$\rho=4\cos\phi$



NOTE: $\theta=0..2\pi$

b) $\boxed{V=\int_0^{2\pi}\int_0^{\arccos 1/4}\int_1^{4\cos\phi} 1\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$

c) $\boxed{\overset{\text{Maple}}{=}\ \frac{81\pi}{8}}$

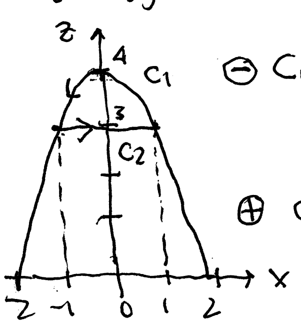
d) $V_s=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos\phi} 1\cdot\rho^2\sin\phi\, d\phi\, d\theta=\frac{32}{3}\pi$

$\left(=\frac{4}{3}\pi(2)^3\right)$

$\frac{V}{V_s}=\left(\frac{81\pi}{8}\right)\left(\frac{3}{32\pi}\right)=\frac{243}{256}\approx 0.949\sim\boxed{95\%}$

② $\vec{F}=\langle -x^2y, xy\rangle$

$\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=\frac{\partial}{\partial x}(xy)-\frac{\partial}{\partial y}(-x^2y)=y+x^2$



⊖ $C_1: x=t, y=4-t^2$

$\vec{r}=\langle t, 4-t^2\rangle\quad t=-1..1$

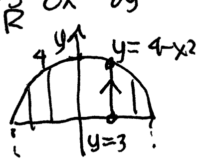
$\vec{r}\,'=\langle 1, -2t\rangle$

⊕ $C_2: x=t, y=3$

$\vec{r}=\langle t, 3\rangle\quad t=-1..1$

$\vec{r}\,'=\langle 1, 0\rangle$

$\iint_R \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\, dA=\int_{-1}^{1}\int_3^{4-x^2} y+x^2\, dy\, dx \overset{\text{Maple}}{=}\boxed{\frac{24}{5}}$



② continued $F=\langle -x^2y, xy\rangle$

$C_1: \vec{F}(\vec{r}(t))=\langle -(t)^2(4-t^2), t(4-t^2)\rangle$

$=\langle t^4-4t^2, 4t-t^3\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)=1(t^4-4t^2)+(-2t)(4t-t^3)$

$=t^4-4t^2-8t^2+2t^3=t^4-12t^2+2t^4$

$\int_{C_1}\vec{F}\cdot d\vec{r}=-\int_{-1}^{1}3t^4-12t^2\, dt \overset{\text{Maple}}{=}\frac{34}{5}$

reverse

$C_2: \vec{F}(\vec{r}(t))=\langle -(t)^2\cdot 3, t\cdot 3\rangle=\langle -3t^2, 3t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)=1(-3t^2)+0(3t)=-3t^2$

$\int_{C_2}\vec{F}\cdot d\vec{r}=+\int_{-1}^{1}-3t^2\, dt=-t^3\Big|_{-1}^{1}=-2$

$\oint_C \vec{F}\cdot d\vec{r}=\frac{34}{5}-2=\boxed{\frac{24}{5}}$ ✓

③ a) $\vec{r}=\langle t, t^2, t^3\rangle$

$\vec{r}\,'=\langle 1, 2t, 3t^2\rangle$

$\vec{F}=\langle yz, zx, xy\rangle$

$\vec{F}(\vec{r}(t))=\langle (t^2)(t^3), (t^3)(t), t(t^2)\rangle$

$=\langle t^5, t^4, t^3\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)=1(t^5)+2t(t^4)+3t^2(t^3)$

$=t^5+2t^5+3t^5=6t^5$

$\int_C \vec{F}\cdot dr=\int_{\frac{1}{2}}^{1}6t^5\, dt=t^6\Big|_{1/2}^{1}=1-\frac{1}{2^6}$

$=\boxed{\frac{63}{64}}$

b) $\cancel{\vec{r}(\tfrac{1}{2})=\langle -\tfrac{1}{2}, \tfrac{1}{4}, -\tfrac{1}{8}\rangle}$ straight line segment evaluation

$\cancel{\vec{r}(1)=\langle 1,1,1\rangle}$ so

$\cancel{\vec{r}(1)-\vec{r}(\tfrac{1}{2})=\langle 1+\tfrac{1}{2}, 1-\tfrac{1}{4}, 1+\tfrac{1}{8}\rangle=\langle \tfrac{3}{2}, \tfrac{3}{4}, \tfrac{9}{8}\rangle}$

$\cancel{\vec{r}=\langle \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}\rangle+t\langle \tfrac{3}{2}, \tfrac{3}{4}, \tfrac{9}{8}\rangle}$

$\cancel{=\langle \tfrac{1}{2}(1+3t), \tfrac{1}{4}(1+3t), \tfrac{1}{8}(1+9t)\rangle}$

$\cancel{\vec{r}\,'=\langle \tfrac{3}{2}, \tfrac{3}{4}, \tfrac{9}{8}\rangle}$

$\cancel{\vec{F}(\vec{r}(t))=\langle \tfrac{1}{32}(1+3t)(1+9t), \tfrac{1}{16}(1+t)(1+9t), \tfrac{1}{8}(1+t)(1+3t)\rangle}$

NAH! mercy.

b) $\vec{\nabla}\times\vec{F}=\left\langle \underset{123}{\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}}, \underset{231}{\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}}, \underset{312}{\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}}\right\rangle$

$=\left\langle \frac{\partial}{\partial y}(xy)-\frac{\partial}{\partial z}(zx), \frac{\partial}{\partial z}(yz)-\frac{\partial}{\partial x}(xy), \frac{\partial}{\partial x}(zx)-\frac{\partial}{\partial y}(yz)\right\rangle$

$=\langle x-x, y-y, z-z\rangle=\langle 0,0,0\rangle$

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③ continued

c) $\langle yz, zx, xy\rangle = \vec{F} = \vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$

$\int \left[\frac{\partial f}{\partial x} = yz\right] dx \rightarrow \quad f = xyz + C(y,z)$

$\frac{\partial f}{\partial y} = zx \longrightarrow \quad \frac{\partial f}{\partial y} = \cancel{xz} + \frac{\partial C}{\partial y}(y,z) = \cancel{zx}$

$\frac{\partial C}{\partial y}(y,z) = 0$

$C = C(z)$

$f = xyz + C(z)$

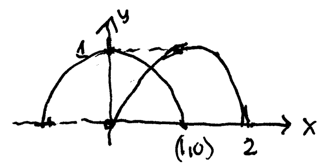
$\frac{\partial f}{\partial z} = xy \longrightarrow \quad \frac{\partial f}{\partial z} = \cancel{xy} + C'(z) = \cancel{xy}\ 0$

$C'(z) = 0 \rightarrow C(z) = k$

can set $k=0 \longrightarrow \quad \boxed{f = xyz + k}$

e) $\int_C \vec{F}\cdot d\vec{r} = f(1,1,1) - f(-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8})$

$= 1 - \frac{1}{64} = \boxed{\frac{63}{64}}$ ✓



intersection of circles:

$x^2+y^2=1$

$x^2+y^2=2x$

subtract: $0 = 1-2x \rightarrow x = \frac{1}{2} \rightarrow y = \frac{\sqrt{3}}{2}$

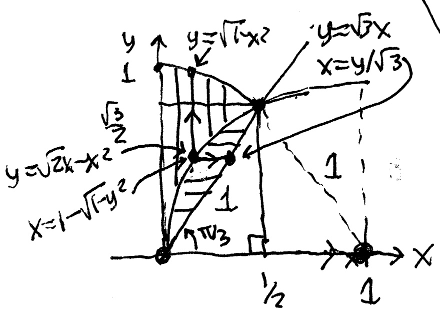
$y = \sqrt{3}x$

also on line

④ $\int_{y=0}^{y=\frac{\sqrt{3}}{2}} \int_{x=1-\sqrt{1-y^2}}^{x=y/\sqrt{3}} xy\, dx\, dy + \int_{x=0}^{x=\frac{1}{2}} \int_{y=\sqrt{2x-x^2}}^{y=\sqrt{1-x^2}} xy\, dy\, dx$

start on right circle move right to diagonal line

start on right circle, move up to left circle.



$y=\sqrt{1-x^2} \rightarrow y^2 = 1-x^2$

$x^2+y^2=1$ unit circle at origin

$y=\sqrt{2x-x^2} \rightarrow y^2 = 2x-x^2$

$x^2+y^2=2x \rightarrow (x-1)^2-1+y^2=0 \quad (x-1)^2+y^2=1$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$ — circle radius 1 at (1,0)

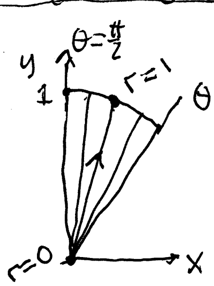
$x = 1-\sqrt{1-y^2}$

$(x-1)^2 = 1-y^2$

$(x-1)^2+y^2=1$ — same

so $r = 0..1,\ \theta = \pi/3..\pi/2$

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^1 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta$



$= \underbrace{\int_0^1 r^3\, dr}_{\frac{r^4}{4}\Big|_0^1 = \frac{1}{4}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \underbrace{\sin\theta}_{u}\ \underbrace{\cos\theta\, d\theta}_{du} = \boxed{\frac{1}{32}}$

$u^2/2 = \frac{\sin^2\theta}{2}$

$\frac{\sin^2\theta}{2}\Big|_{\frac{\pi}{3}}^{\pi/2} = \frac{1}{2}\left(1-\sin^2\frac{\pi}{3}\right) = \frac{1}{2}\left(1-\underbrace{\left(\frac{\sqrt{3}}{2}\right)^2}_{3/4}\right)$

$= \frac{1}{8}$