MAT2705-02/04 16F Quiz 7 Print Name (Last, First) ______________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (**but give decimal approximations for interpretation**). Indicate where technology is used and what type (Maple, GC).

1. $4\,y''(t)+4\,y'(t)+5\,y(t)=2\cos\left(\frac{t}{2}\right)$, $y(0)=0$, $y'(0)=0$ [Maple notation].

a) State Maple's solution of the initial value problem (use function notation $y(t)$).

b) Put the DE into standard linear form. Then identify the values of the damping constant and characteristic time $k_0=1/\tau_0$, the natural frequency ω_0, and the quality factor $Q=\omega_0\,\tau_0$, exactly and numerically. Is this underdamped, critically damped or overdamped?

c) Find the general solution by hand, showing all steps.

d) Find the solution satisfying the initial conditions, showing all steps.

e) Give exact and numerical values of the amplitude and phase shift of the steady state solution (particular solution!) and re-express the sinusoidal factor of this solution in phase-shifted cosine form. [Make sure you use a diagram to justify your values.] State what numerical fraction of a cycle (2π) the phase shift is (i.e., evaluate $\delta/2\pi$) as well as its numerical value in degrees, and whether the cosine curve is shifted left (earlier in time) or right (later in time) on the time line (by a phase less than or equal to half a cycle of course). Explain.

f) Find the two envelope functions of the decaying oscillating transient solution.

[Optional: Make a rough sketch of the plot of this transient and its envelope on an appropriate decay window.]

g) Make a rough sketch of the plot of your full solution with the steady state solution until they merge.

steady state

transient

▶ **solution**

① a) $\boxed{y(t) = -\frac{2}{5}e^{-\frac{t}{2}}\cos t - \frac{3}{10}e^{-\frac{t}{2}}\sin t + \frac{2}{5}\cos\frac{t}{2} + \frac{1}{5}\sin\frac{t}{2}}$

$\begin{bmatrix}4&2\\-2&4\end{bmatrix}\begin{bmatrix}c_3\\c_4\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$ $\begin{bmatrix}c_3\\c_4\end{bmatrix}=\frac{1}{16+4}\begin{bmatrix}4&-2\\2&4\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4\\2\end{bmatrix}=\frac{1}{5}\begin{bmatrix}2\\1\end{bmatrix}$

$y_p=\frac{1}{5}(2\cos\frac{t}{2}+\sin\frac{t}{2})$ (steady state soln)

$\boxed{y=y_h+y_p=e^{-\frac{t}{2}}(c_1\cos t+c_2\sin t)+\frac{1}{5}(2\cos\frac{t}{2}+\sin\frac{t}{2})}$

d) $y'=-\frac{1}{2}e^{-\frac{t}{2}}(c_1\cos t+c_2\sin t)+e^{-\frac{t}{2}}(-c_1\sin t+c_2\cos t)$
$+\frac{1}{10}(-2\sin\frac{t}{2}+\cos\frac{t}{2})$

$y(0)=c_1+\frac{2}{5}=0\to c_1=-\frac{2}{5}$

$y'(0)=-\frac{1}{2}c_1+c_2+\frac{1}{10}=0\to c_2=-\frac{1}{10}-\frac{1}{2}\left(\frac{2}{5}\right)=-\frac{3}{10}$

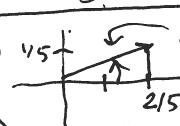
$\boxed{y=e^{-\frac{t}{2}}\left(-\frac{2}{5}\cos t-\frac{3}{10}\sin t\right)+\frac{1}{5}\left(2\cos\frac{t}{2}+\sin\frac{t}{2}\right)}$ yay!

b) $y''(t)+y'(t)+\frac{5}{4}y(t)=\frac{1}{2}\cos\frac{t}{2}$

$k_0=1$ $\omega_0=\frac{\sqrt{5}}{2}\approx 1.12$

$\tau_0=1$ $Q=\omega_0\tau_0=\frac{\sqrt{5}}{2}\approx1.12>\frac{1}{2}$

underdamped

e) 

$A=\frac{\sqrt{5}}{5}\approx0.447$ steady state soln.

$\delta=\arctan\frac{1}{2}\approx0.464>0$

$\approx26.6^\circ$

$\frac{\delta}{2\pi}\approx0.074$ cycles

peaks to right — later in time

c) $(4D^2+4D+5)y=2\cos\frac{t}{2}\to r=\pm\frac{1}{2}i$

$(D^2+\frac{1}{4})(2\cos\frac{t}{2})=0$

$y=e^{rt}\to 4r^2+4r+5=0$

Maple: $r=-\frac{1}{2}\pm i$

$e^{rt}=e^{-\frac{1}{2}t}e^{\pm it}=e^{-\frac{t}{2}}(\cos t\pm i\sin t)$

$\{e^{-\frac{t}{2}}\cos t, e^{-\frac{t}{2}}\sin t\}$

$y_h=e^{-\frac{t}{2}}(c_1\cos t+c_2\sin t)$ (transient)

no root interference

transient (3-4-5 triangle!)

f) $A_0=\frac{1}{10}\sqrt{4^2+3^2}=\frac{5}{10}=\frac{1}{2}=0.5$

$y=\pm\frac{1}{2}e^{-\frac{t}{2}}$ are envelope functions, see next page

g) see next page

choose transient decay window ($\tau=2$)

$t=0..\underbrace{5(2)}_{10}$

$5\,[y_p=c_3\cos\frac{t}{2}+c_4\sin\frac{t}{2}]$

$4\,[y_p'=-\frac{1}{2}c_3\sin\frac{t}{2}+\frac{1}{2}c_4\cos\frac{t}{2}]$

$+\,[y_p''=-\frac{1}{4}c_3\cos\frac{t}{2}-\frac{1}{4}c_4\sin\frac{t}{2}]$

$4y_p''+4y_p'+5y_p=\underbrace{[(5-1)c_3+2c_4]}_{=2}\cos\frac{t}{2}+\underbrace{[-2c_3+(5-1)c_4]}_{=0}\sin\frac{t}{2}=2\cos\frac{t}{2}$

e) $\boxed{y_{ss}=\frac{\sqrt{5}}{5}\cos\left(\frac{t}{2}-\arctan\frac{1}{2}\right)}$ oops

Okay, I somehow forgot to scan page 2 with my hand drawn graph taken from this plot.

g) the decay window is $t = 0\,..5 \cdot 2 = 10$

> $plot\left(\left[-\frac{3}{10}\,e^{-\frac{1}{2}t}\sin(t) - \frac{2}{5}\,e^{-\frac{1}{2}t}\cos(t) + \frac{1}{5}\sin\left(\frac{1}{2}t\right) + \frac{2}{5}\cos\left(\frac{1}{2}t\right), \frac{1}{5}\sin\left(\frac{1}{2}t\right) + \frac{2}{5}\cos\left(\frac{1}{2}t\right)\right], t = 0\,..10, color = [red, blue]\right)$