Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). You are encouraged to use technology to check all of your hand results. You may use Maple for row reduction without showing individual steps and for matrix inverses.

1. A coupled system of ODEs representing a 2 mass 3 spring system $\left(m_1 = 1 = m_2, k_1 = 6, k_2 = 2, k_3 = 3\right)$ has the following equations of motion and initial conditions:
$x_1''(t) = -8\ x_1(t) + 2\,x_2(t), x_2''(t) = 2\,x_1(t) - 5\,x_2(t) + 35\cos(t),$
$x_1(0) = 5, x_2(0) = 0,\ x_1'(0) = 0, x_2'(0) = 5$.

a) Solve this system with Maple and write down the solutions for the two unknowns. [Use function notation for all variables in Maple: $x1(t), x1''(t)$, *etc.*] If you typed it in correctly, your solution will satisfy $x_1(\pi) = -37/3$.

b) Identify the values of the two natural frequencies $\omega_1 < \omega_2$ which appear in these solutions together with the driving frequency $\omega_3 = 1$. What are the values of the corresponding 3 periods T_1, T_2, T_3? What is the common period T of the three oscillations? How many cycles of each of these fit into that full period?

c) Rewrite this system of DEs and its initial conditions in explicit matrix form $\vec{x}'' = A\,\vec{x} + \vec{F}$ for the vector variable $\vec{x} = \langle x_1, x_2\rangle$, identifying the coefficient matrix A and the driving vector $\vec{F}$.

d) Use Maple to write down its choice of eigenvalues and eigenvectors of A.

e) By hand showing all steps, find the **smallest integer component** eigenvectors $\vec{b}_1, \vec{b}_2$ of the coefficient matrix A produced by the solution algorithm after rescaling of the standard results by positive multiples if necessary, ordered so that the corresponding eigenvalues satisfy $|\lambda_1| < |\lambda_2|$. Evaluate the matrix $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$ and its inverse, and the diagonalized matrix $A_B = B^{-1} A\, B$.
[Use technology to check that your inverse is correct and make sure your results agree with Maple's eigenvectors modulo rescaling and/or permutation.]

f) What are the slopes m_1, m_2 of the lines through the origin containing the two eigenvectors? On the grid provided, draw in those two lines, labeling them by their corresponding coordinates y_1, y_2 in the positive direction determined by the eigenvectors and then indicate by thicker arrows both eigenvectors $\vec{b}_1, \vec{b}_2$, labeled by their symbols. Recall $\vec{x} = B\,\vec{y}, \vec{y} = B^{-1}\,\vec{x}$, where $\vec{y} = \langle y_1, y_2\rangle$. Also label the x_1, x_2 axes.

g) Evaluate $\vec{y}(0) = B^{-1}\,\vec{x}(0), \vec{y}'(0) = B^{-1}\,\vec{x}'(0), B^{-1}\,\vec{F}$ to find the new components of these three vectors.

h) On the grid provided, draw in the vectors $\vec{x}(0)$ and $\vec{x}'(0)$ and the new axes of the coordinates $\{y_1, y_2\}$ together with their unit tickmarks and label these vectors and all 4 axes properly. On your graph, draw in exactly the parallelograms parallel to the new coordinate axes which projects this vectors along those axes and lightly shade them in in pencil (pen?). Are the part g) components consistent with your plot? Explain.

i) Find by hand the general solution of the corresponding decoupled system of DEs $\vec{y}'' = A_B\vec{y} + B^{-1}\vec{F}$.
First write these equations out in explicit matrix form, then obtain the two equivalent scalar DEs which are its components. Then solve them to find their general solutions using the method of undetermined coefficients. State your general solutions in scalar form and box them: $y_1(t) = ..., y_2(t) = ...$, identifying the homogeneous and particular parts of each solution: $y_1 = y_{1h} + y_{1p}, y_2 = y_{2h} + y_{2p}$.

j) Then express the general solution for $\vec{x} = B\,\vec{y}$ in explicit matrix form (without multiplying matrix factors) ***and impose the initial conditions*** using matrix methods to solve the linear systems. Write out and box the final scalar solutions: $x_1(t) = ..., x_2(t) = ...$. Do they agree with Maple's solution from part a)? If not, look for your error. Did you input the equations correctly?

k) Express the (correct) solution as a sum of the two eigenvector modes and the response mode in the form: $\vec{x} = y_{1h}\,\vec{b}_1 + y_{2h}\,\vec{b}_2 + \cos(t)\ \vec{b}_3$ thus identifying the particular solution $\vec{x}_p$ (last term), the response vector coefficient $\vec{\boldsymbol{b}}_3$ and the homogeneous solution $\vec{\boldsymbol{x}}_h$ (first two terms), as well as the final expressions for the two decoupled variables y_{1h} and y_{2h}. Which homogeneous term is associated with the tandem mode and which with the accordian mode? Is the response term a tandem or accordian mode?

l) Write each of these sinusoidal functions y_{1h} and y_{2h} in phase-shifted cosine form stating explicitly (A_1, δ_1) and (A_2, δ_2) respectively, making a single completely labeled diagram in the sinusoidal coefficient plane that supports your work for each case. Evaluate the two vectors $A_1 \vec{\boldsymbol{b}}_1$ and $A_2 \vec{\boldsymbol{b}}_2$ as well as $\vec{\boldsymbol{b}}_3$ numerically to 2 decimal places.

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have not accessed any of the class web pages or any other sites during the exam. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: