MAT2705-01/02 16F Takehome Test 3 Answers(1)

① a) $4x''+4x'+17x = 17te^{-t} \rightarrow \begin{array}{l} r=-1 \\ m=2 \end{array}$ $(D+1)^2(17te^{-t})=0$ no root overlap

$x_h = e^{rt} \hookrightarrow \underbrace{(4r^2+4r+17)}_{=0}e^{rt}=0$

$r = \frac{-4\pm\sqrt{16-4\cdot4\cdot17}}{2\cdot4} = -\frac{1\pm\sqrt{-16}}{2} = \frac{-1\pm4i}{2}$

$x_h = e^{(-\frac12\pm2i)t} = e^{-\frac t2}(\cos 2t \pm i\sin 2t)$

$\hookrightarrow x_h = e^{-t/2}(c_1\cos 2t + c_2 \sin 2t)$

$17\,[x_p = (c_3+c_4t)e^{-t}]$

$4\,[x_p' = (c_4-(c_3+c_4t))e^{-t}]$

$4\,[x_p'' = (-c_4-(c_4-c_3-c_4t))e^{-t}]$

$4x_p''+4x_p'+17x_p = \left[\begin{array}{l} 17(c_3+c_4t) \\ +\ 4(c_4-c_3-c_4t) \\ +\ 4(-2c_4+c_3+c_4t)\end{array}\right]e^{-t}$

$=[(17-4+4)c_3 + (4-8)c_4 + (17-4+4)c_4t]e^{-t}$

$= (\underbrace{(17c_3-4c_4)}_{=0} + \underbrace{17c_4}_{=17\rightarrow c_4=1} t)e^{-t} = 17te^{-t}$

$\hookrightarrow c_3 = \frac{4}{17}c_4 = \frac{4}{17}$ $\qquad y_p = (\frac{4}{17}+t)e^{-t}$

$x = x_h + x_p = e^{-t/2}(c_1\cos 2t + c_2\sin 2t) + (\frac{4}{17}+t)e^{-t}$

$x' = -\frac12 e^{-t/2}(c_1\cos 2t + c_2\sin 2t) + 2e^{-t/2}(-c_1\sin 2t + c_2\cos 2t) + [-(\frac{4}{17}+t)+1]e^{-t}$

$x(0) = c_1 + \frac{4}{17} = 0 \rightarrow c_1 = -\frac{4}{17}$

$x'(0) = -\frac12 c_1 + 2c_2 + \frac{13}{17} = 0 \rightarrow c_2 = \frac12\left(\frac12\left(-\frac{4}{17}\right) - \frac{13}{17}\right) = -\frac{15}{34}$

$$\boxed{x = e^{-t/2}\left(-\frac{4}{17}\cos 2t - \frac{15}{34}\sin 2t\right) + \left(\frac{4}{17}+t\right)e^{-t}}$$

$0 = x' = -\frac12 e^{-t/2}(-\frac{4}{17}\cos 2t - \frac{15}{34}\sin 2t) + 2e^{-t/2}(\frac{4}{17}\sin 2t - \frac{15}{34}\cos 2t) + (-\frac{4}{17} - t + 1)e^{-t}$

$= e^{-t/2}\left(\frac{2-15}{17}\cos 2t + \left(\frac{2}{17}+\frac{15}{4\cdot17}\right)\sin 2t\right) + \left(\frac{13}{17}-t\right)e^{-t}$

$= e^{-t/2}\left(-\frac{13}{17}\cos 2t + \frac{23}{68}\sin 2t\right) + \left(\frac{13}{17}-t\right)e^{-t}$ must be solved numerically.

From plot ~~maximum~~ extremum occurs near $t=2$: $\boxed{t \approx 1.7805,\ x \approx 0.5018}$

b) $4x''+4x'+16x = 4(3\cos\omega t + 4\sin\omega t) = 4\cdot5\cos(\omega t - \arctan 4/3)$

$x''+x'+4x = 5\cos(\omega t - \arctan 4/3)$

$k_0=1$, $\boxed{\tau_0=1}$ $\qquad \omega_0^2=4$, $\boxed{\omega_0=2}$ $\qquad \boxed{Q=\omega_0\tau_0=2}$ $\qquad \boxed{T_0=\frac{2\pi}{\omega_0}=\pi}\boxed{\approx 3.142}$

x_h: $4r^2+4r+16=0$, $r=\frac{-4\pm\sqrt{16-4\cdot4\cdot16}}{2\cdot4} = -\frac12 \pm \frac{\sqrt{15}}{2}i$

$\boxed{k_1 = -\frac12 = -0.5,\ \omega_0 = \frac{\sqrt{15}}{2} \approx 1.936}$

$16\,[x_p = c_3\cos\omega t + c_4\sin\omega t]$

$4\,[x_p' = -\omega c_3\sin\omega t + \omega c_4\cos\omega t]$

$4\,[x_p'' = -\omega^2 c_3\cos\omega t - \omega^2 c_4\sin\omega t]$

$4x_p''+4x_p'+16x_p = [(16-4\omega^2)c_3 + 4\omega c_4]\cos\omega t$
$+ [-4\omega c_3 + (16-4\omega^2)]\sin\omega t$
$= 12\cos\omega t + 16\sin\omega t$

$\begin{bmatrix} (16-4\omega^2) & 4\omega \\ -4\omega & (16-4\omega^2)\end{bmatrix}\begin{bmatrix} c_3 \\ c_4\end{bmatrix} = \begin{bmatrix} 12 \\ 16\end{bmatrix}$

$\begin{bmatrix} c_3 \\ c_4\end{bmatrix} = \frac{\begin{bmatrix} 16-4\omega^2 & -4\omega \\ 4\omega & 16-4\omega^2\end{bmatrix}\begin{bmatrix} 12 \\ 16\end{bmatrix}}{(16-4\omega^2)^2+16\omega^2} = \frac{16\begin{bmatrix} 4-\omega^2 & -\omega \\ \omega & 4-\omega^2\end{bmatrix}\begin{bmatrix} 3 \\ 4\end{bmatrix}}{16[(4-\omega^2)^2+\omega^2]}$

$= \frac{\begin{bmatrix} 12-4\omega-3\omega^2 \\ 16+3\omega-4\omega^2\end{bmatrix}}{\omega^4-7\omega^2+16}$

$$\boxed{x_p = \frac{(12-4\omega-3\omega^2)\cos\omega t + (16+3\omega-4\omega^2)\sin\omega t}{\omega^4-7\omega^2+16}}$$

Maple agrees.

MAT2705-01/02 16F Takehome Test 3 Answers (2)

① b) continued

$$A(\omega)=\frac{\sqrt{(12-4\omega+3\omega^2)^2+(16-3\omega+4\omega^2)^2}}{\omega^4-7\omega^2+16}$$

Maple

$$=\frac{5\sqrt{\omega^4-7\omega^2+16}}{\omega^4-7\omega^2+16}=\boxed{\frac{5}{\sqrt{\omega^4-7\omega^2+16}}=A(\omega)}$$

plotwindow $\omega=0..12$ seems appropriate.
peak near natural frequency $\omega_0=2$.

$$0=A'(\omega)=-\frac{5}{2}(\omega^4-7\omega^2+16)^{-3/2}\underbrace{(4\omega^3-14\omega)}_{2\omega(2\omega^2-7)}$$

$$\hookrightarrow \omega=0,\ \underbrace{\boxed{\sqrt{\tfrac{7}{2}}\approx 1.8708}}_{\omega_p}$$

$$A(\omega_p)=\frac{5}{\sqrt{\left(\frac{7}{2}\right)^2-7\left(\frac{7}{2}\right)+16}}=\frac{5}{\sqrt{\frac{49-98+64}{4}}}=\frac{5}{\sqrt{\frac{15}{4}}}=\boxed{\frac{2}{3}\sqrt{15}\approx 2.582}$$

$$A(0)=\frac{5}{\sqrt{16}}=\frac{5}{4},\quad \frac{A(\omega_p)}{A(0)}=\frac{8}{5}\frac{\sqrt{15}}{3}=\boxed{\frac{8}{\sqrt{15}}\approx 2.066}$$

looks right in plot.

compare to $Q=2$, close!

② a) $\boxed{\langle x_1,x_2\rangle=\langle e^{-3t}(12\sin 2t+5\cos 2t),\ e^{-3t}(2\sin 2t+3\cos 2t)\rangle}$

b) $$\begin{bmatrix}x_1'\\x_2'\end{bmatrix}=\underbrace{\begin{bmatrix}-6&13\\-1&0\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\end{bmatrix},\quad \begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}$$

c) $$0=|A-\lambda I|=\begin{vmatrix}-6-\lambda&13\\-1&-\lambda\end{vmatrix}=\lambda(\lambda+6)+13=\lambda^2+6\lambda+13$$

$$\hookrightarrow \lambda=\frac{-6\pm\sqrt{36-4\cdot 13}}{2}=\frac{-6\pm 4i}{2}=-3\pm 2i$$

$\lambda=-3\pm 2i$:

$$A+(3-2i)I=\begin{bmatrix}-6-(-3+2i)&13\\-1&-(-3+2i)\end{bmatrix}=\begin{bmatrix}-3-2i&13\\-1&3-2i\end{bmatrix}$$

$$\xrightarrow{\text{rref}}\begin{bmatrix}1&-3+2i\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

L F

$x_2=t$
$x_1+(-3+2i)t=0$
$x_1=(3-2i)t$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}(3-2i)t\\t\end{bmatrix}=t\underbrace{\begin{bmatrix}3-2i\\1\end{bmatrix}}_{\vec b_+}\quad \vec b_-=\overline{\vec b_+}=\begin{bmatrix}3+2i\\1\end{bmatrix}$$

$$\boxed{B=\begin{bmatrix}3-2i&3+2i\\1&1\end{bmatrix}}\leftarrow\boxed{\vec b_\pm=\begin{bmatrix}3\mp 2i\\1\end{bmatrix}}$$

Maple agrees.

② d) $\vec x=B\vec y,\ \vec y=B^{-1}\vec x,\ A_B=B^{-1}AB=\begin{bmatrix}-3-2i&0\\0&3+2i\end{bmatrix}$

$\vec x\,'=A\vec x$

$\vec y\,'=A_B\vec y$

$$\begin{bmatrix}y_1'\\y_2'\end{bmatrix}=\begin{bmatrix}-3+2i&0\\0&-3-2i\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}(-3+2i)\,y_1\\(-3-2i)\,y_2\end{bmatrix}$$

$y_1'=(-3+2i)y_1\rightarrow y_1=c_1e^{(-3+2i)t}$

$y_2'=(-3-2i)y_2\rightarrow y_2=c_2e^{(-3-2i)t}\rightarrow c_2=\bar c_1$

$$\vec x=c_1\underbrace{e^{-3t}e^{+2it}\vec b_+}+\text{c.c.}$$

$$\vec z_+=e^{-3t}(\cos 2t+i\sin 2t)\begin{bmatrix}3-2i\\1\end{bmatrix}$$

$$=e^{-3t}\begin{bmatrix}3\cos 2t+2\sin 2t+i(-2\cos 2t+3\sin 2t)\\\cos 2t+i\sin 2t\end{bmatrix}$$

$$=\underbrace{e^{-3t}\begin{bmatrix}3\cos 2t+2\sin 2t\\\cos 2t\end{bmatrix}}_{\vec X_1}+i\underbrace{e^{-3t}\begin{bmatrix}-2\cos 2t+3\sin 2t\\\sin 2t\end{bmatrix}}_{\vec X_2}$$

$$\boxed{\begin{bmatrix}x_1\\x_2\end{bmatrix}=c_1\vec X_1+c_2\vec X_2=c_1e^{-3t}\begin{bmatrix}3\cos 2t+2\sin 2t\\\cos 2t\end{bmatrix}+c_2e^{-3t}\begin{bmatrix}-2\cos 2t+3\sin 2t\\\sin 2t\end{bmatrix}}$$

e) $$\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}=c_1\begin{bmatrix}3\\1\end{bmatrix}+c_2\begin{bmatrix}-2\\0\end{bmatrix}=\begin{bmatrix}3c_1-2c_2\\c_1\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}$$

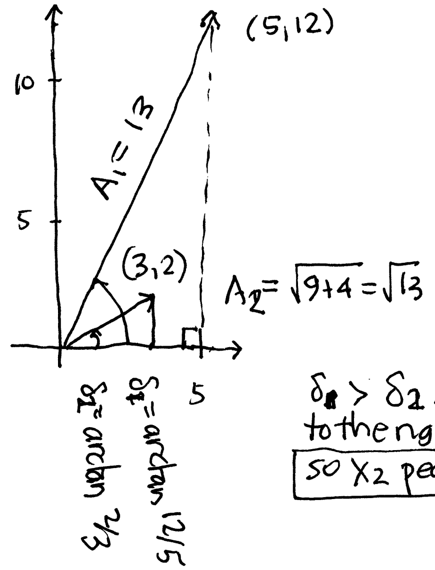
$c_1=3,\ c_2=\frac{1}{2}(3(3)-5)=+2$

$$\boxed{\begin{bmatrix}x_1\\x_2\end{bmatrix}=e^{-3t}\begin{bmatrix}3(3\cos 2t+2\sin 2t)+2(-2\cos 2t+3\sin 2t)\\3\cos 2t+2\sin 2t\end{bmatrix}=e^{-3t}\begin{bmatrix}5\cos 2t+12\sin 2t\\3\cos 2t+2\sin 2t\end{bmatrix}}$$

Maple agrees.

MAT2705-01/02 16F Take home Test 3 Answers (3)

(2) f)



$\delta_1 > \delta_2$ so x_1 shifted more to the right compared to x_2

$\boxed{\text{so } x_2 \text{ peaks to the left of } x_1}$

envelopes:

$$\boxed{\begin{aligned} x_1 &= 13e^{-3t}\cos(2t - \arctan 12/5) \to x_1 = \pm 13 e^{-3t} \\ x_2 &= \sqrt{13}e^{-3t}\cos(2t - \arctan 2/3) \to x_2 = \pm\sqrt{13}e^{-3t} \end{aligned}}$$

g) $\tau = 3 \to 5\tau = 15$ so plot $t = 0..15$

x_1 touches upper envelope to the right of x_2 as predicted

(3) a) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

$$0 = |A - \lambda I| = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda+3)(\lambda+2) - 2 = \lambda^2 + 5\lambda + \overset{6-2}{\overset{\shortparallel}{4}}$$

$$\lambda = \frac{-5 \pm \sqrt{25 - 16}}{2} = \frac{-5 \pm 3}{2} = -4, -1 \qquad \lambda_2 < \lambda_1$$

$$\lambda_1 = -1: A + I = \begin{bmatrix} -3+1 & 1 \\ 2 & -2+1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \to \underset{\text{LF}}{\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = t$, $x_1 = \frac{1}{2}x_2 = \frac{1}{2}t$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/2 \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ $\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\lambda = -4: A + 4I = \begin{bmatrix} -3+4 & 1 \\ 2 & -2+4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \to \underset{\text{LF}}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = t$, $x_1 = -x_2 = -t$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\vec{b}_2$

$$\boxed{B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}} \qquad A_B = B^{-1}AB = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

(3b) $B^{-1} = \frac{1}{1+2}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = B^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4+5 \\ -8+5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

c) $\vec{x} = B\vec{y}$, $\vec{y} = B^{-1}\vec{x} \longrightarrow$

$\vec{y}\,' = A_B\vec{y}$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -y_1 \\ -4y_2 \end{bmatrix}$$

$y_1' = -y_1$ $\quad y_1 = c_1 e^{-t}$

$y_2' = -4y_2$ $\quad y_2 = c_2 e^{-4t}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-4t} \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3e^{-t} \\ -e^{-4t} \end{bmatrix} = 3e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - e^{-4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3e^{-t} + e^{-4t} \\ 6e^{-t} - e^{-4t} \end{bmatrix}}$$

d) The parallelogram projection sides indeed are $3\vec{b}_1$ and $-\vec{b}_2$ and the arrows line up along the new coord axes with directions to and from the origin corresponding to the eigenvalue signs.

e) $\tau_1 = 1 > \tau_2 = \frac{1}{4}$, $5\tau_1 = 5$

$$\langle x_1(5), x_2(5)\rangle = \langle 3e^{-5} + e^{-20}, 6e^{-5} - e^{-20}\rangle$$

$$\approx \langle 0.0202, 0.0404\rangle$$