MAT2705-01/02 16F Take Home Test 3 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions, matrix inverses, plotting and root finding without showing intermediate steps.** ***Print*** the requested technology plots, labeling them and **annotating them appropriately *by hand*** and attach to the end of your test. All differential equations should be solved "by hand" unless otherwise specified.

1. The displacement $x(t)$ of a damped harmonic oscillator system satisfies

$$m\,x''(t) + c\,x'(t) + K\,x(t) = F(t).$$

a) Let $m=4$, $c=4$, $K=17$, $F(t) = 17\,t\,e^{-t}$ and the initial conditions $x(0)=0$, $x'(0)=0$ describe the action of an extended impulse driving force on the system initially at rest. Evaluate the initial value problem solution by hand and use calculus to determine the largest displacement from equilibrium (maximize $|x(t)|$) and when it occurs, to 4 decimal place accuracy. Plot your solution for an appropriate decay window starting at $t=0$ and extending to show the asymptotic behavior, confirming your claim for the critical point and labeling that point [**plot#1**].

b) Let $m=4$, $c=4$, $K=16$, $F(t) = 4\,(3\cos(\omega t) + 4\sin(\omega t))$, $\omega \geq 0$ and evaluate the steady state solution by hand to explore resonance.

What are the natural frequency ω_0, natural decay time τ_0, the quality factor $Q=\omega_0\tau_0$ and the natural period $T_0 = 2\pi/\omega_0$ for this system? What are the transient solution frequency ω_1 and decay constant τ_1? (Give both exact and numeric values to 3 decimal places.) Evaluate and **plot** the (simplified!) amplitude $A(\omega)$ of the steady state solution for nonnegative frequency, showing it reaching its asymptotic value at large frequency [**plot#2**]. Use calculus to find the exact critical point at which the resonance peak occurs. Evaluate also the ratio $A(\omega_{peak})/A(0)$.

2. $x_1'(t) = -6\,x_1(t) + 13\,x_2(t)$, $x_2'(t) = -\,x_1(t)$, $x_1(0) = 5$, $x_2(0) = 3$

a) Write down the Maple solution of this initial value problem, simpified to integer coefficients.

b) Rewrite this system of DEs **and** its initial conditions explicitly in matrix form for the vector variable $\boldsymbol{x} = \langle x_1, x_2\rangle$ as a column matrix (using the actual matrices, not their symbols), identifying the coefficient matrix A.

c) Derive by hand its eigenvalues $\lambda_\pm$ and eigenvectors $\vec{\boldsymbol{b}}_\pm$, $B = \langle \vec{\boldsymbol{b}}_+ | \vec{\boldsymbol{b}}_- \rangle$, and check that they agree with Maple.

d) Find the general solution of the DE system.

e) Find the solution of the initial value problem.

f) Express the sinusoidal factor in each vector component of $\vec{\boldsymbol{x}}$ in phase-shifted form $x_i = A_i\,e^{-kt}\cos(\omega t - \delta_i)$ to identify the exponential envelope functions. Back up your calculations with a common single diagram in the coefficient plane with the coefficient vectors of the two sinusoidal functions. Based on comparing the two phase shifts, which variable has its peaks shifted to the left of the other, x_1 or x_2?

g) Plot x_1 and x_2 versus t (use the original expressions, not the phase-shifted ones) for 5 characteristic times of the exponential factor starting at $t=0$, including the envelopes of both decaying oscillations [**plot#3**]. Is your claim in the previous part reflected in the plot? Do your solution curves touch their envelopes?

3. $x_1'(t) = -3\,x_1(t) + 1\,x_2(t)$, $x_2'(t) = 2\,x_1(t) - 2\,x_2(t)$, $x_1(0) = 4$, $x_2(0) = 5$.

a) Identify the coefficient matrix and by hand find a new basis for R^2 consisting of eigenvectors $\vec{\boldsymbol{b}}_1$, $\vec{\boldsymbol{b}}_2$ of this matrix using the standard hand recipe, rescaling them to be the **minimal integer** eigenvectors. Order the real eigenvalues $\lambda_1 \geq \lambda_2$ by decreasing value (increasing absolute value). Identify $B = \langle \vec{\boldsymbol{b}}_1 | \vec{\boldsymbol{b}}_2 \rangle$.

b) Evaluate the new coordinates $\langle y_1, y_2\rangle$ of the point $\langle x_1, x_2\rangle = \langle 4, 5\rangle$ with respect to this basis of eigenvectors.

c) Solve this initial value problem by hand, showing all steps. Make sure the final result agrees with Maple.

d) Use technology to **plot [#4]** a directionfield for this DE with the solution curve through the single initial data point, and (by hand if necessary) include the lines through the two eigenvectors representing the two subspaces of eigenvectors. Choose an appropriate window that shows everything clearly without wasting additional window space. By hand label these lines by their new coordinate labels, draw in and label the eigenvectors and the initial data vector $\vec{x}(0)$ themselves as arrows, and include the parallelogram projection of the latter vector onto the new coordinate axes, i.e., the parallelogram parallel to the new coordinate axes with the initial data vector as the main diagonal. Do the projections along the coordinate axes agree with the values you found for the new coordinates of this vector? Explain. Does your directionfield correspond to the eigenvectors you have drawn? Explain why.
e) Which of the two characteristic times in this problem is longer and hence determines the timescale on which the asymptotic limit is "reached". What are the exact and numerical values of the two variables at 5 times this characteristic time?

Advice. When in doubt about how much work to show, show more. Explain using words if it helps. Think of this take-home test as an exercise in "writing intensive" technical expression. Try to impress bob as though it were material for a job interview. In a real world technical job, you need to be able to write coherent technical reports that other people can follow.

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have read the long instructions on the class web page. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: