MAT2705-02/05 15S Quiz 10 NOT! Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always SIMPLIFYexpressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. $x_1{}' = x_2, x_2{}' = -13\,x_1 - 4\,x_2, x_1(0) = 0, x_2(0) = 3$

a) Write down the Maple solution of this initial value problem.

b) Rewrite this system of DEs **and** its initial conditions explicitly in matrix form for the vector variable $\boldsymbol{x} = \langle x_1, x_2 \rangle$ as a column matrix (using the actual matrices, not their symbols), identifying the coefficient matrix A.

c) Derive by hand its eigenvalues $\lambda_\pm = -k \pm I\,\omega$ and eigenvectors $\vec{\boldsymbol{b}}_\pm, B = \langle \vec{\boldsymbol{b}}_+ | \vec{\boldsymbol{b}}_- \rangle$, scaling them up to integers if necessary, and check that they agree with Maple.

d) Evaluate the real and imaginary parts of $\vec{z} = e^{\lambda_+ t}\vec{\boldsymbol{b}}_+ = \vec{\boldsymbol{u}} + I\,\vec{\boldsymbol{v}}$.

e) Let $\vec{\boldsymbol{x}} = c_1\,\vec{\boldsymbol{u}} + c_2\,\vec{\boldsymbol{v}}$. Solve the condition $\vec{\boldsymbol{x}}(0) = \langle 0, 3 \rangle$ for (c_1, c_2), backsubstitute into $\vec{\boldsymbol{x}}$ and simplify. Make sure that it agrees with part a).

f) Express the sinusoidal factor in each vector component of $\vec{\boldsymbol{x}}$ in phase-shifted form $x_i = A_i\,e^{-kt}\cos(\omega\,t - \delta_i)$ to identify the exponential envelope functions. Evaluate the associated exponential characteristic time τ and the period T of the oscillation and the ratio $\frac{T}{5\,\tau}$ of the number of periods that fit into a decay window. Based on comparing the two phase shifts, which variable has its peaks shifted to the left of the other, x_1 or x_2?

g) Plot x_1 and x_2 versus t (use the original expressions, not the phase-shifted ones) for 5 characteristic times of the exponential factor starting at $t = 0$, including the envelopes of both decaying oscillations.

▶ solution

1 a) $x_1 = e^{-2t}\sin(3t)\,,\; x_2 = e^{-2t}(3\cos(3t) - 2\sin(3t))$

b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -13 & -4 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\,,\quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

c) $0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -13 & -4-\lambda \end{vmatrix} = \lambda(\lambda+4) + 13$

$= \lambda^2 + 4\lambda + 13 \to \lambda = \dfrac{-4 \pm \sqrt{16 - 4\cdot 13}}{2} = \boxed{-2 \pm 3i = \lambda_\pm}$

$\lambda_+ = -2+3i$:

$A + (2-3i)I = \begin{bmatrix} 2-3i & 1 \\ -13 & -4+2-3i \end{bmatrix} = \begin{bmatrix} 2-3i & 1 \\ -13 & -2-3i \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2+3i}{13} \\ 0 & 0 \end{bmatrix}$

$\begin{array}{cc} \begin{bmatrix} 1 & (2+3i)/3 \\ 0 & 0 \end{bmatrix} \\ \mathrm{L} \quad\quad \mathrm{F} \end{array}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 + \frac{1}{3}(2+3i)x_2 = 0 \quad x_1 = -\frac{1}{3}(2+3i)t$

$x_2 = t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(2+3i)t/3 \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} -(2+3i)/13 \\ 1 \end{bmatrix}}_{\vec{b}_+}$ $\quad \vec{b}_- = \overline{\vec{b}_+}$

$\boxed{B = \begin{bmatrix} (-2-3i)/13 & (-2+3i)/13 \\ 1 & 1 \end{bmatrix}} \longrightarrow \boxed{B = \begin{bmatrix} -2-3i & -2+3i \\ 13 & 13 \end{bmatrix}}$

MAPLE agrees here $\quad$ rescaled to integers

f) $T = \frac{2\pi}{3}\,,\; \tau = \frac{1}{2}\,,\; T/5\tau = \frac{2\pi}{3}\left(\frac{2}{5}\right) = \frac{4\pi}{15} \approx 0.84$

d) $e^{(-2+3i)t}\begin{bmatrix} -2-3i \\ 13 \end{bmatrix} = e^{-2t}(\cos 3t + i\sin 3t)\begin{bmatrix} -2-3i \\ 13 \end{bmatrix}$

$\vec{z} = e^{-2t}\begin{bmatrix} -2\cos 3t + 3\sin 3t + i(-3\cos 3t - 2\sin 3t) \\ 13\cos 3t \quad + i\,13\sin 3t \end{bmatrix}$

$= \underbrace{e^{-2t}\begin{bmatrix} -2\cos 3t + 3\sin 3t \\ 13\cos 3t \end{bmatrix}}_{\vec{u}} + i\,\underbrace{e^{-2t}\begin{bmatrix} -3\cos 3t - 2\sin 3t \\ 13\sin 3t \end{bmatrix}}_{\vec{v}}$

e) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} -2\cos 3t + 3\sin 3t \\ 13\cos 3t \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -3\cos 3t - 2\sin 3t \\ 13\sin 3t \end{bmatrix}$

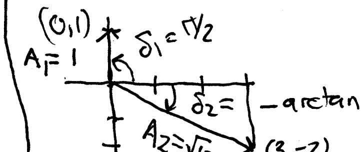
$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -2c_1 - 3c_2 \\ 13c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \to c_1 = 3/13 \longrightarrow c_2 = -\frac{2}{3}\left(\frac{3}{13}\right) = -2/13$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-2t}\left(\frac{3}{13}\begin{bmatrix} -2C + 3S \\ 13C \end{bmatrix} - \frac{2}{13}\begin{bmatrix} -3C - 2S \\ 13S \end{bmatrix}\right)$

$= e^{-2t}\begin{bmatrix} \frac{(-6+6)}{13}C + \frac{(9+4)}{13}S \\ 3C \quad -2S \end{bmatrix} = e^{-2t}\begin{bmatrix} \sin 3t \\ 3\cos 3t - 2\sin 3t \end{bmatrix}$

yeah! agrees with part a)

f)



$-\arctan 2/3$

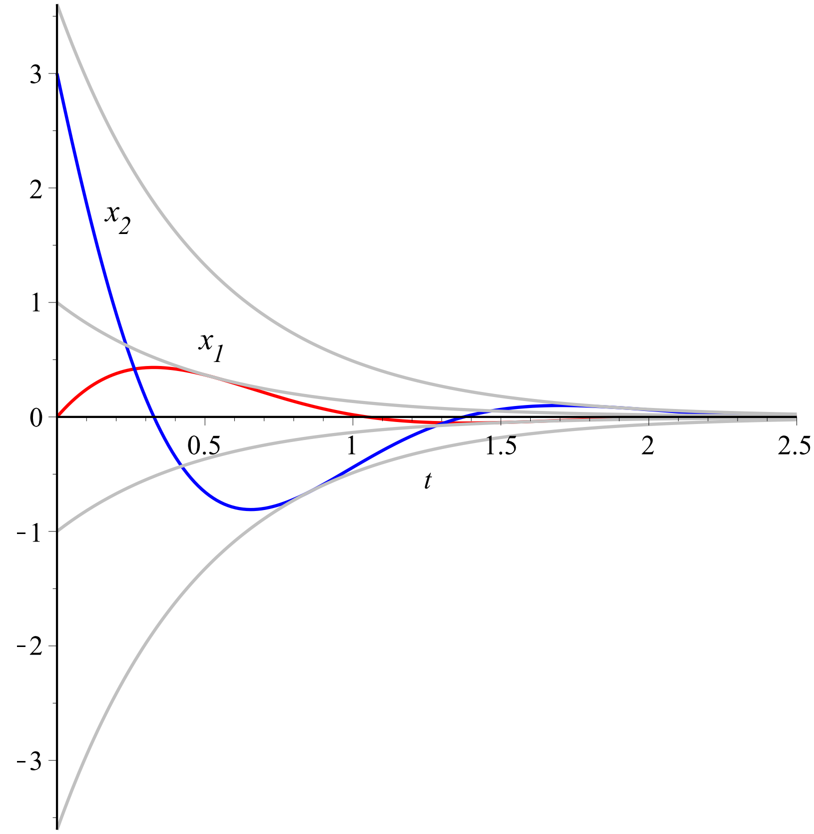
$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-2t}\begin{bmatrix} \cos(3t - \pi/2) \\ \sqrt{13}\cos(3t + \arctan 2/3) \end{bmatrix}}$

x_2 is shifted left, x_1 shifted right

so x_2 is shifted to left of x_1 by an obtuse angle seen in figure $\delta_1 - \delta_2$.

$\left.\begin{array}{l} x_1: \pm e^{-2t} \\ x_2: \pm\sqrt{13}e^{-2t} \end{array}\right\}$ envelopes

> $plot\left(\left[e^{-2t}\sin(3\,t),\ e^{-2t}\,(3\cos(3\,t)-2\sin(3\,t)),\ \sqrt{13}\ e^{-2t}, -\sqrt{13}\ e^{-2t},\ e^{-2t}, -\ e^{-2t}\right], t=0\,..\,\frac{5}{2}, color = [red, blue, gray\$4]\right)$



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