MAT2705-02/05 15S Quiz 8b NOT! Print Name (Last, First) ____________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (**but give decimal approximations for interpretation**). Indicate where technology is used and what type (Maple, GC).

1. $my''+cy'+ky= \sin(\omega t)$; $m=4$, $c=4, k=37$. [prime is d/dt]

a) State Maple's general solution (use function notation $y(t)$) and identify the steady state solution (the sinusoidal term in the solution).

b) Now derive this steady state solution using the method of undetermined coefficients.

c) Evaluate the amplitude $A(\omega)$ of this steady state solution, simplifying the final formula.

d) Use calculus to find exactly the single critical point $(\omega_p, A(\omega_p))$ of this function for $\omega > 0$. Check that it is correct with Maple. Approximate it to 3 decimal places.

e) Evaluate the ratio $\dfrac{A(\omega_p)}{A(0)}$ and compare it to the quality factor Q .

f) Sketch a technology plot of the amplitude function in a window of width showing well both the resonance peak and the approach to the horizontal axis. Check that your maximum in the plot agrees with your calculated location.

▶ solution

① a) $4y''+4y'+37y = \sin\omega t$

MAPLE:

$$\boxed{y = \underbrace{e^{-t/2}(c_1\cos 3t + c_2\sin 3t)}_{y_h} + \underbrace{\frac{-4\omega\cos\omega t + (37-4\omega^2)\sin\omega t}{16\omega^4-280\omega^2+1369}}_{y_{ss}=y_p}}$$

b)
$$37\,[y_p = c_3\cos\omega t + c_4\sin\omega t]$$
$$4\,[y_p' = -\omega c_3\sin\omega t + \omega c_4\cos\omega t]$$
$$4\,[y_p'' = -\omega^2 c_3\cos\omega t - \omega^2 c_4\sin\omega t]$$

$$4y_p''+4y_p'+37y_p = [\underbrace{(37-4\omega^2)c_3 + 4\omega c_4}_{=0}]\cos\omega t + [\underbrace{-4\omega c_3 + (37-4\omega^2)c_4}_{=4}]\sin\omega t = 4\sin\omega t$$

$$(37-4\omega^2)c_3 + 4\omega c_4 = 0$$
$$-4\omega c_3 + (37-4\omega^2)c_4 = 1$$

$$\begin{bmatrix} 37-4\omega^2 & 4\omega \\ -4\omega & 37-4\omega^2 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4\end{bmatrix} = \begin{bmatrix} 0 \\ 1\end{bmatrix}$$

$$\begin{bmatrix} c_3 \\ c_4\end{bmatrix} = \frac{1}{(37-4\omega^2)^2+16\omega^2}\begin{bmatrix} 37-4\omega^2 & -4\omega \\ 4\omega & 37-4\omega^2 \end{bmatrix}\begin{bmatrix} 0 \\ 1\end{bmatrix}$$
$$= \frac{1}{\underbrace{(37-4\omega^2)^2+16\omega^2}_{=16\omega^4-280\omega^2+1369}}\begin{bmatrix} -4\omega \\ 37-4\omega^2\end{bmatrix}$$

$$\boxed{y_p = \frac{-4\omega\cos\omega t + (37-4\omega^2)\sin\omega t}{(37-4\omega^2)^2+16\omega^2}}$$

c)
$$A(\omega) = \frac{\sqrt{16\omega^2+(37-4\omega^2)^2}}{(37-4\omega^2)^2+16\omega^2} = \frac{1}{\sqrt{(37-4\omega^2)^2+16\omega^2}} = \frac{1}{\sqrt{16\omega^4-280\omega^2+1369}}$$

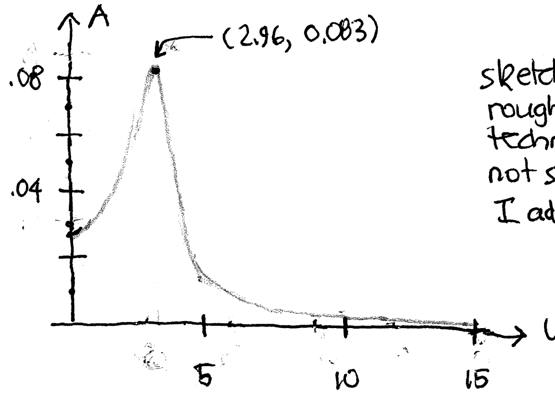
d)
$$0 = A'(\omega) = -\tfrac12(\cdots)^{-1/2}(4\cdot 16\omega^3 - 2\cdot 280\omega)$$
$$\hookrightarrow 4\omega^3 - 35\omega = 4\omega(\omega^2 - \tfrac{35}{4}) = 0$$
$$\omega = \sqrt{\tfrac{35}{4}} = \boxed{\tfrac{\sqrt{35}}{2} = \omega_p \approx 2.958}$$
$$A(\omega_p) = [(37-4\cdot(\tfrac{35}{4}))^2 + 16\cdot(\tfrac{35}{4})]^{-1/2}$$
$$= (4+4\cdot 35)^{-1/2} = (4\cdot 36)^{-1/2} = \frac{1}{2\cdot 6} = \boxed{\frac{1}{12}}$$
$$\approx \boxed{0.083}$$

e)
$$A(0) = \frac{1}{37} \approx 0.0270$$
$$\frac{A(\omega_p)}{A(0)} = \frac{1/12}{1/37} = \frac{37}{12} \approx \boxed{3.083}$$
pretty close

recall $k_0 = 1$, $\tau_0 = 1$, $\underbrace{\omega_0 = \sqrt{37}/2 \approx 3.041}$, $Q = \omega_0\tau_0 \approx \boxed{3.041}$

(a little bigger than ω_p)

f)



sketching a graph roughly from technology is not so easy I admit.