MAT 2705-02/03 15S Final Exam Answers (1)

① a) Maple soln:

$$x_1 = 4\sin\tfrac{t}{2} + \cos\tfrac{t}{3} + 2\cos\tfrac{2}{3}t - 3\sin\tfrac{2}{3}t$$
$$x_2 = \underbrace{-4\sin\tfrac{t}{2}}\ \underbrace{+2\cos\tfrac{t}{3}}\ \underbrace{-2\cos\tfrac{2}{3}t + 3\sin\tfrac{2}{3}t}$$

b) $\omega_3 = \frac12 \qquad \omega_1 = \frac13 \qquad \omega_2 = \frac23$

$T_3 = 2\pi/(1/2) = 4\pi \qquad T_1 = 2\pi/(1/3) = 6\pi \qquad T_2 = 2\pi/(2/3) = 3\pi$

$12\pi = 3T_3 = 2T_1 = 4T_2$ ← # cycles of each in 12π

c) Divide by 9:

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \underbrace{\begin{bmatrix} -3/9 & 1/9 \\ 2/9 & -2/9 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 7/9 \sin t/2 \\ -7/9 \sin t/2 \end{bmatrix}}_{\vec F}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

d) Eigenvectors(A): $\underbrace{\begin{bmatrix} -4/9 \\ -1/9 \end{bmatrix}}_{\lambda_{Maple}}, \underbrace{\begin{bmatrix} -1 & 1/2 \\ 1 & 1 \end{bmatrix}}_{B_{Maple}}$

e) $0 = |A - \lambda I| = \begin{vmatrix} -\frac39-\lambda & \frac19 \\ \frac29 & -\frac29-\lambda \end{vmatrix}$

$= (\lambda+\frac13)(\lambda+\frac29) - \frac{2}{81}$

$= \lambda^2 + \frac59\lambda + \frac{2\cdot3}{27\cdot3} - \frac{2}{81}$

$= \lambda^2 + \frac59\lambda + \frac{4}{81} = 0 \xrightarrow{Maple} \lambda = \underbrace{-\frac19}_{\lambda_1}, \underbrace{-\frac49}_{\lambda_2}$

$|\lambda_1| < |\lambda_2|$:

$\lambda = -\frac19$: $A + \frac19 I = \begin{bmatrix} \frac19-\frac39 & \frac19 \\ \frac29 & \frac19-\frac29 \end{bmatrix} = \begin{bmatrix} -\frac29 & \frac19 \\ \frac29 & -\frac19 \end{bmatrix}$

$$\xrightarrow{rref} \begin{array}{cc} L & F \end{array} \begin{bmatrix} 1 & -\frac12 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_1 - \tfrac12 x_2 = 0$$

$x_2 = t$, $x_1 = \frac12 t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac12 t \\ t \end{bmatrix} = t\begin{bmatrix} \frac12 \\ 1 \end{bmatrix} = \frac t2 \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec b_1}$$

double for integer entries

$\lambda = -\frac49$: $A + \frac49 I = \begin{bmatrix} \frac49-\frac39 & \frac19 \\ \frac29 & \frac49-\frac29 \end{bmatrix} = \begin{bmatrix} \frac19 & \frac19 \\ \frac29 & \frac29 \end{bmatrix}$

$$\xrightarrow{rref} \begin{array}{cc} L & F \end{array} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_1 + x_2 = 0 \quad x_1 = -t$$

$x_2 = t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec b_2}$$

e) continued: $\lambda = -1/9, -4/9$

$B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ agrees with Maple

$B^{-1} = \frac13\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, \quad A_B = B^{-1}AB = \begin{bmatrix} -1/9 & 0 \\ 0 & -4/9 \end{bmatrix}$

f) $\vec b_1 = \langle 1, 2\rangle \quad m_1 = 2/1 = 2$

$\vec b_2 = \langle -1, 1\rangle \quad m_2 = 1/-1 = -1$

g) $\vec y(0) = B^{-1}\vec x(0) = \frac13\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$

$B^{-1}F = \frac13\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\left(\frac79 \sin\frac t2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

$= (\sin\frac t2)\frac{7}{27}\begin{bmatrix} 1-1 \\ -2-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -7/9 \end{bmatrix}(\sin\frac t2) = \begin{bmatrix} 0 \\ -\frac79 \sin\frac{t}{2} \end{bmatrix}$

oops! eye slipped.

h) $\vec x(0) = 1\vec b_1 - 2\vec b_2$ see graph

i) $\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -1/9 & 0 \\ 0 & -4/9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac79 \sin\frac t2 \end{bmatrix} = \begin{bmatrix} -\frac{y_1}{9} \\ -\frac{4y_2}{9} - \frac79\sin\frac t2 \end{bmatrix}$

$y_1'' + \frac19 y_1 = 0 \qquad \to y_{1h} = c_1\cos\frac t3 + c_2 \sin\frac t3$

$y_2'' + \frac49 y_2 = -\frac79 \sin\frac t9 \to y_{2h} = c_3\cos\frac{2t}{3} + c_4\sin\frac23 t$

$\frac49\,[y_{2p} = c_5\cos\frac t2 + c_6 \sin\frac t2]$

$1\,[y_{2p}'' = -\frac14 c_5 \cos\frac t2 - \frac14 c_6 \sin\frac t2]$

$y_{2p}'' + \frac49 y_{2p} = \underbrace{(\frac49 - \frac14)}_{\frac{7}{36}} c_5 \cos\frac t2 + \underbrace{(\frac49-\frac14)}_{7/36} c_6 \sin\frac t9 = -\frac79 \sin\frac t2$

$\frac{7}{36}c_5 = 0 \qquad \frac{7}{36}c_6 = -\frac79$

$c_5 = 0 \qquad c_6 = \frac{36}{7}(-\frac79) = -4$

$$\boxed{\begin{aligned} y_1 &= y_{1h} = c_1\cos\tfrac t3 + c_2\sin\tfrac t3 \qquad (y_{1p} = 0) \\ y_2 &= y_{2h} + y_{2p} = c_3\cos\tfrac23 t + c_4\sin\tfrac{2t}{3} - 4\sin\tfrac t2 \end{aligned}}$$

j) $\vec x = (c_1\cos\frac t3 + c_2\sin\frac t3)\vec b_1 + (c_3\cos\frac23 t + c_4\sin\frac{2t}{3} - 4\sin\frac t2)\vec b_2$

$\vec x(0) = c_1\vec b_1 + c_3\vec b_2 = B\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = B^{-1}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\vec x' = (-\frac13 c_1 \sin\frac t3 + \frac13 c_2\cos\frac t3)\vec b_1 + (-\frac23 c_3 \sin\frac{2t}{3} + \frac23 c_4\cos\frac{2t}{3} - \frac42\cos\frac t2)\vec b_2$

$\vec x'(0) = \frac13 c_2 \vec b_1 + (\frac23 c_4 - \frac12)\vec b_2 = \vec 0$

$= B\begin{bmatrix} c_2/3 \\ 2c_4/3 - 2 \end{bmatrix} \to \begin{bmatrix} c_2/3 \\ 2c_4/3 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$c_2 = 0, \quad c_4 = 3$

$\boxed{\begin{aligned} y_{1h} &= \cos t/3 \\ y_{2h} &= -2\cos\tfrac{2t}{3} + 3\sin 2t/3 \end{aligned}}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cos\tfrac t3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-2\cos\tfrac{2t}{3} + 3\sin\tfrac{2t}{3} - 4\sin\tfrac t2)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \underbrace{\cos\tfrac t3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{tandem}} + \underbrace{(-2\cos\tfrac{2t}{3} + 3\sin\tfrac{2t}{3})\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{accordian}} + \sin\tfrac t2 \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

accordian: $\vec b_3$

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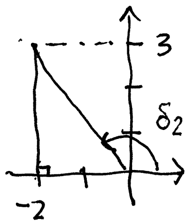
k) $y_{1h} = \cos\frac{t}{3} \to \langle 1,0\rangle \to A_1 = 1,\ \delta_1 = 0$ $\to$ already in phase shifted cosine form!

$y_{2h} = -2\cos\frac{2t}{3} + 3\sin\frac{2t}{3} \to \langle -2,3\rangle$



2nd quadrant

$\delta_2 = \pi - \arctan 3/2 \approx 2.1588\,(\text{rad})$

$A = \sqrt{2^2+3^2} = \sqrt{13}$

$$\boxed{y_{2h} = \sqrt{13}\cos\left(\frac{2t}{3} - \pi + \arctan\frac{3}{2}\right)}$$

j) continued: scalar solutions

$$\boxed{\begin{aligned} x_1 &= \cos\frac{t}{3} + 2\cos\frac{2t}{3} - 3\sin\frac{2t}{3} + 4\sin\frac{t}{2} \\ x_2 &= 2\cos\frac{t}{3} - 2\cos\frac{2t}{3} + 3\sin\frac{2t}{3} - 4\sin\frac{t}{2} \end{aligned}}$$

agrees with Maple!

Note: even without knowledge of eigenvectors, this is easily rewritten:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\frac{t}{3} + 2\cos\frac{2t}{3} - 3\sin\frac{2t}{3} + 4\sin t/2 \\ 2\cos\frac{t}{3} - 2\cos\frac{2t}{3} + 3\sin\frac{2t}{3} - 4\sin t/3 \end{bmatrix}$$

$$= \underbrace{\cos\frac{t}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(2\cos\frac{2t}{3} - 3\sin\frac{2t}{3}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\sin\frac{t}{2}\begin{bmatrix} 4 \\ -4 \end{bmatrix}}_{\vec{x}_p}$$

$\vec{x}_h$, vector coefficients of functions must be proportional to any choice of eigenvectors (eigenvectors only defined to within a scalar multiple)

vector functions correspond to decoupled variables

clearly $\vec{x}_p$ response to $\omega_3 = \frac{1}{2}$ driving term

eigenvector components) and which with the accordian mode (opposite signed eigenvector components)? Is the response term tandem or accordian?
l) Write each of these sinusoidal functions y_{1h} and y_{2h} in phase-shifted cosine form stating explicitly (A_1, δ_1) and (A_2, δ_2) respectively, making a completely labeled diagram in the sinusoidal coefficient plane that supports your work for each case.
optional (ignore, this is for future students):
m) If you plot one cycle of the two solution functions $t = 0..T$, you see a global maximum departure from the origin (i.e., maximum absolute value) in one of the two variables. Identify this variable and its approximate extreme value and the value of t at which it occurs to 3 decimal places.

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have not accessed any of the class web pages or any other sites during the exam. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date:

b)