

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). You are encouraged to use technology to check all of your hand results. You may use Maple for row reduction without showing individual steps and for matrix inverses.

1. A coupled system of ODEs representing a 2 mass 3 spring system has the following equations of motion  $9x_1'' = -3x_1 + x_2 + 7\sin(t/2)$ ,  $9x_2'' = 2x_1 - 2x_2 - 7\sin(t/2)$ ,  $x_1(0) = 3$ ,  $x_2(0) = 0$ ,  $x_1'(0) = 0$ ,  $x_2'(0) = 0$
- a) Solve this system with Maple and write down the solutions for the two unknowns. [Use function notation for all variables in Maple:  $x1(t)$ ,  $x1''(t)$ , etc.] If you typed it in correctly, you should only have single digit integer coefficients of the elementary functions which appear.
- b) Identify the values of the two natural frequencies  $\omega_1 < \omega_2$  which appear in these solutions together with the driving frequency  $\omega_3 = \frac{1}{2}$ . What are the values of the corresponding 3 periods  $T_1, T_2, T_3$ ? Note that the common period of the three oscillations is  $T = 12\pi$ . How many cycles of each of these fit into that full period?
- c) Rewrite this system of DEs and its initial conditions in explicit matrix form  $\vec{x}'' = A\vec{x} + \vec{F}$  for the vector variable  $\vec{x} = \langle x_1, x_2 \rangle$ , identifying the coefficient matrix  $A$  and the driving vector  $\vec{F}$  (don't forget to divide through by 9!).
- d) Use Maple to write down its choice of eigenvalues and eigenvectors of  $A$ .
- e) By hand showing all steps, find the smallest integer component eigenvectors  $\vec{b}_1, \vec{b}_2$  of the coefficient matrix  $A$  produced by the solution algorithm after rescaling of the standard results by positive multiples if necessary, ordered so that the corresponding eigenvalues satisfy  $|\lambda_1| < |\lambda_2|$ . Evaluate the matrix  $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$  and its inverse, and the diagonalized matrix  $A_B = B^{-1}AB$ . [Use technology to check that your inverse is correct and make sure your results agree with Maple's eigenvectors modulo rescaling and/or permutation.]
- f) What are the slopes  $m_1, m_2$  of the lines through the origin containing the two eigenvectors? On the grid provided, draw in those two lines, labeling them by their corresponding coordinates  $y_1, y_2$  in the positive direction determined by the eigenvectors and then indicate by thicker arrows both eigenvectors  $\vec{b}_1, \vec{b}_2$ , labeled by their symbols. Recall  $\vec{x} = B\vec{y}$ ,  $\vec{y} = B^{-1}\vec{x}$ , where  $\vec{y} = \langle y_1, y_2 \rangle$ . Also label the  $x_1, x_2$  axes.
- g) Evaluate  $\vec{y}(0) = B^{-1}\vec{x}(0)$ ,  $B^{-1}\vec{F}$  to find the new components of these two vectors.
- h) On the grid provided, draw in the vector  $\vec{x}(0)$  and the new axes of the coordinates  $\{y_1, y_2\}$  and label this vector and all 4 axes properly. On your graph, draw in exactly the parallelogram parallel to the new coordinate axes which projects this vector along those axes. Are the part g) components consistent with your plot? Explain.
- i) Find by hand the general solution of the corresponding decoupled system of DEs  $\vec{y}'' = A_B\vec{y} + B^{-1}\vec{F}$ . First write these equations out in explicit matrix form, then obtain the two equivalent scalar DEs which are its components. Then solve them to find their general solutions using the method of undetermined coefficients. State your general solutions in scalar form and box them:  $y_1(t) = \dots, y_2(t) = \dots$ , identifying the homogeneous and particular parts of each solution:  $y_1 = y_{1h} + y_{1p}, y_2 = y_{2h} + y_{2p}$ .
- j) Then express the general solution for  $\vec{x} = B\vec{y}$  in explicit matrix form (without multiplying matrix factors) **and impose the initial conditions** using matrix methods to solve the linear systems. Write out and box the final scalar solutions:  $x_1(t) = \dots, x_2(t) = \dots$ . Do they agree with Maple's solution from part a)? If not, look for your error. Did you input the equations correctly?
- k) Express the (correct) solution as a sum of the two eigenvector modes and the response mode in the form:  $\vec{x} = y_{1h}\vec{b}_1 + y_{2h}\vec{b}_2 + \sin(t/2)\vec{b}_3$  thus identifying the particular solution  $\vec{x}_p$  (last term), the response vector coefficient  $\vec{b}_3$  and the homogeneous solution  $\vec{x}_h$  (first two terms), as well as the final expressions for the two decoupled variables  $y_{1h}$  and  $y_{2h}$ . Which homogeneous term is associated with the tandem mode (same signed

eigenvector components) and which with the accordian mode (opposite signed eigenvector components)? Is the response term tandem or accordian?

l) Write each of these sinusoidal functions  $y_{1h}$  and  $y_{2h}$  in phase-shifted cosine form stating explicitly  $(A_1, \delta_1)$  and  $(A_2, \delta_2)$  respectively, making a completely labeled diagram in the sinusoidal coefficient plane that supports your work for each case.

**optional (ignore, this is for future students):**

m) If you plot one cycle of the two solution functions  $t = 0 .. T$ , you see a global maximum departure from the origin (i.e., maximum absolute value) in one of the two variables. Identify this variable and its approximate extreme value and the value of  $t$  at which it occurs to 3 decimal places.

## ► solution

## ▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:  
"During this examination, all work has been my own. I have not accessed any of the class web pages or any other sites during the exam. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature:

Date:

