

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions, matrix inverses, plotting and root finding without showing intermediate steps.** *Print* the requested technology plots, labeling them and **annotating them appropriately by hand** and attach to the end of your test. All differential equations should be solved "by hand" unless otherwise specified.

1. The displacement $x(t)$ of an underdamped harmonic oscillator system satisfies

$$m x''(t) + c x'(t) + K x(t) = F(t).$$

Let $m = 9$, $c = 6$, $K = 10$ and the initial conditions $x(0) = x_0$, $x'(0) = v_0$.

a) Express this DE in standard form with a unit coefficient of the second derivative term. What are the natural frequency ω_0 , natural decay time τ_0 , the quality factor $Q = \omega_0 \tau_0$ and the period $T_0 = 2\pi/\omega_0$ for this system? (Give both exact and numeric values to 3 decimal places.)

Consider the following driving force functions $F(t)$:

b) $F(t) = 0$. Find the general solution of the differential equation. What is the frequency ω_1 , decay time τ_1 and the period $T_1 = 2\pi/\omega_1$ for this decaying sinusoidal solution? (Give both exact and numeric values to 3 decimal places.)

c) $F(t) = 0$. Solve the above initial value problem for $x_0 = 3$, $v_0 = -3$. Evaluate the two envelope functions for the exponentially decaying sinusoidal function and **plot (#1)** all three together for $t = 0 \dots 0.5 \tau_1$, fully labeling the axes and three curves. [You should see a number of oscillations before convergence to the horizontal axis.]

d) $F(t) = 17(e^{-t} - e^{-2t})$, $x_0 = 0$, $v_0 = 0$. Find the solution of this initial value problem and then find the maximum displacement from equilibrium and the time at which it occurs numerically to 4 decimal places. **Plot (#2)** your solution together with $\frac{F(t)}{17}$ for $t = 0 \dots 0.5 \tau_1$. Compare your coordinate values to your graph, annotating the maximal point on the graph.

e) $F(t) = 17 m \sin(\omega t)$, $\omega > 0$, $x_0 = 0$, $v_0 = 0$.

Find the steady state solution by hand, where the frequency ω of the driving force function is a parameter.

f) Evaluate the steady state amplitude function $A(\omega)$ and its zero frequency value $A(0)$.

g) Use calculus to find the exact and numerical values of both the frequency ω_p and the amplitude ratio $A(\omega_p)/A(0)$ where it has its peak value for $\omega \geq 0$. What is the numerical value of $A(\omega_p)/A(0)$ and how does its value compare to the quality factor Q ?

h) **Plot (#3)** the amplitude function ratio $A(\omega)/A(0)$ and its derivative in an appropriate window (showing the limiting behavior of the entire function for $\omega \geq 0$ and hand annotate on your axes the values of ω_p , ω_0).

i) $F(t) = 17 m \sin(2t)$, $x_0 = 0$, $v_0 = 0$. Use Maple to find the solution of the initial value problem. Identify the steady state solution. Then evaluate the values of the amplitude A and phase shift δ for the steady state solution (the part of the solution which remains after the transient has died away). Evaluate the relative phase shift $\delta_{rel} = \delta - \frac{\pi}{2} + 2\pi$ compared to the sine function (the extra cycle is necessary to put it into the correct interval of values for this case). Express this relative phase shift in radians, degrees and cycles (divide radians by 2π).

f) Make a single **plot (#4)** in an appropriate viewing window ($t = 0 \dots 0.5 \tau_1$) showing both the solution function and the steady state solution until they merge, together with the rescaled driving function $A \sin(2t)$. Identify all three by hand annotations. Mark off the interval between two consecutive peaks of the latter function with vertical lines to the peaks and connect those lines above the peaks with a horizontal line, then mark off on that line the intermediate peak of the steady state solution in between. Is that intermediate peak lagging behind in time as it should, and does the fraction of the period you see agree with $\delta_{rel}/2\pi$? Explain. [Finally check that your solution curve starts out at the origin of coordinates. Does it?]

2. $x_1'(t) = -k_1 x_1(t) + k_3 x_3(t)$, $x_2'(t) = k_1 x_1(t) - k_2 x_2(t)$, $x_3'(t) = k_2 x_2(t) - k_3 x_3(t)$,
 $x_1(0) = 45$, $x_2(0) = 0$, $x_3(0) = 45$; $k_i = r/V_i$, $r = 18$, $(V_1, V_2, V_3) = (3, 6, 6)$.

- If 90 units of stuff are equally distributed in concentration among these 3 tanks, what is the common concentration and how much stuff is there in each tank?
- Write this closed 3 tank mixing tank system of differential equations for the vector variable $\vec{x} = \langle x_1, x_2, x_3 \rangle$ AND its initial conditions in explicit matrix form (showing all components, not just vector symbols) and identifying the coefficient matrix A .
- Use the eigenvector approach to find its general solution by hand, showing all steps.
- Find the IVP solution, using matrix methods showing all steps. Make sure it agrees with Maple's solution.
- Evaluate the asymptotic solution \vec{x}_∞ for $t \rightarrow \infty$, which is the equilibrium solution approached by your solution.
- What is the common exponential characteristic time τ for the decay of the concentration differences?
- Plot (#5)** the three solution curves together with their horizontal asymptotes in an appropriate viewing window ($t = 0 \dots 5\tau$) that shows them reaching and settling down to those equilibrium values without compressing the interesting part before reaching that limit. Label each curve. **Replot (#6)** the corresponding 3 concentrations. Does their common asymptotic value agree with part a)?

3. $x_1'(t) = -22 x_1(t) + 12 x_2(t)$, $x_2'(t) = 3 x_1(t) - 13 x_2(t)$, $x_1(0) = -3$, $x_2(0) = 7$.

- Identify the coefficient matrix and by hand find a new basis for R^2 consisting of eigenvectors \vec{b}_1, \vec{b}_2 of this matrix using the standard hand recipe, rescaling them to be the **minimal integer** eigenvectors. Order the real eigenvalues $\lambda_1 \geq \lambda_2$ by decreasing value (increasing absolute value). Identify $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$.
- Evaluate the new coordinates $\langle y_1, y_2 \rangle$ of the point $\langle x_1, x_2 \rangle = \langle -3, 7 \rangle$ with respect to this basis of eigenvectors.
- Solve this initial value problem by hand, showing all steps. Make sure the final result agrees with Maple.
- Use technology to **plot (#7)** a directionfield for this DE with the solution curve through the single initial data point, and (by hand if necessary) include the lines through the two eigenvectors representing the two subspaces of eigenvectors. Choose an appropriate window that shows everything clearly without wasting additional window space. By hand label these lines by their new coordinate labels, draw in and label the eigenvectors and the initial data vector $\vec{x}(0)$ themselves as arrows, and include the parallelogram projection of the latter vector onto the new coordinate axes, i.e., the parallelogram parallel to the new coordinate axes with the initial data vector as the main diagonal. Do the projections along the coordinate axes agree with the values you found for the new coordinates of this vector? Explain. Does your directionfield correspond to the eigenvectors you have drawn? Explain why.
- Which of the two characteristic times in this problem is longer and hence determines the timescale on which the asymptotic limit is "reached". What are the exact and numerical values of the two variables at 5 times this characteristic time?

Advice. When in doubt about how much work to show, show more. Explain using words if it helps. Think of this take-home test as an exercise in "writing intensive" technical expression. Try to impress bob as though it were material for a job interview. In a real world technical job, you need to be able to write coherent technical reports that other people can follow.

► solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
 "During this examination, all work has been my own. I have read the long instructions on the class web page. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature:

Date: