MAT2705-02/05 15S Test 2 Print Name (Last, First) ______________________________|__

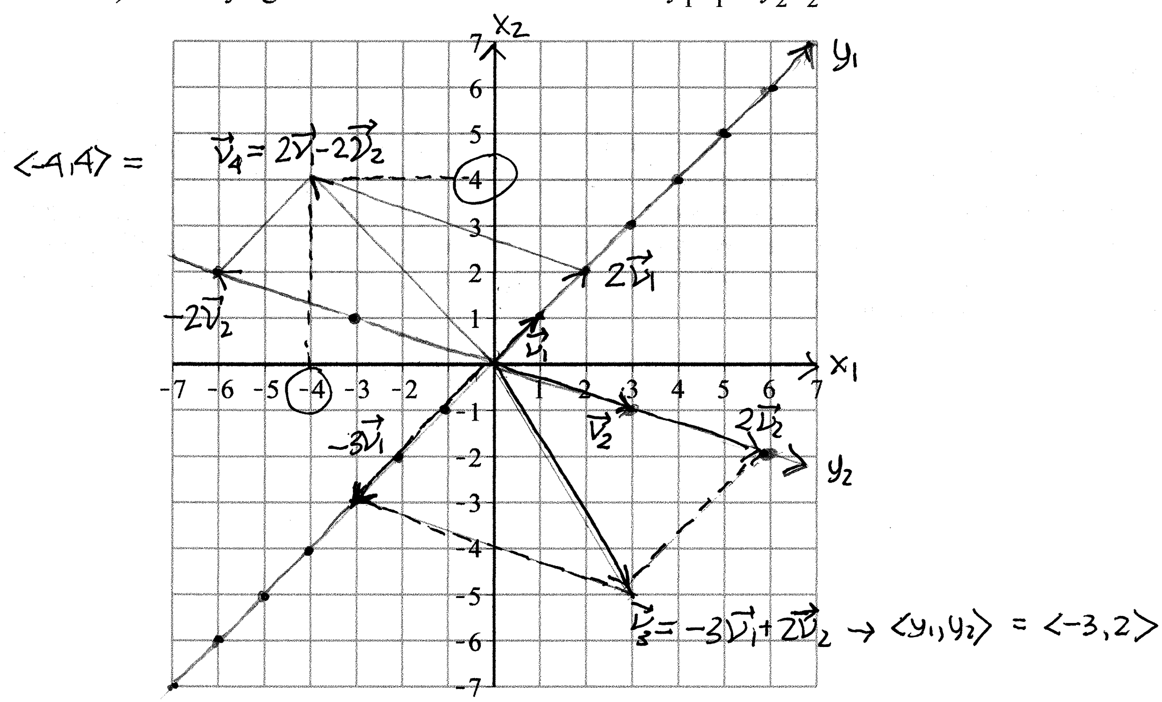
Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **You may use technology for row reductions, determinants and matrix inverses.**

1. a) On the grid below, **draw in** arrows representing the vectors $\vec{v}_1 = \langle 1, 1\rangle$ and $\vec{v}_2 = \langle 3, -1\rangle$ and $\vec{v}_3 = \langle 3, -5\rangle$ and **label** them by their symbols. **Extend** the basis vectors $\{\vec{v}_1, \vec{v}_2\}$ to the corresponding coordinate axes for (y_1, y_2) and **mark** the positive direction with an arrow head and the axis label. Mark off tickmarks on these axes for integer values of the new coordinates. Then **draw in** the parallelogram that graphically expresses $\vec{v}_3$ as a linear combination of $\{\vec{v}_1, \vec{v}_2\}$. **Label** its two sides along the new coordinate axes by the corresponding vectors they represent. Then read off the coordinates (y_1, y_2) of $\vec{v}_3$ with respect to these two vectors (write them down) and **express** $\vec{v}_3$ as a linear combination of these vectors; **put this equation** at the tip of this vector.

b) Now write down the matrix equation that enables you to express $\vec{v}_3$ as a linear combination of the other two vectors, solve that system using the inverse coefficient matrix (stating its value and showing the matrix multiplication and simplification steps), and then express $\vec{v}_3$ explicitly as a linear combination of those vectors, writing $\vec{v}_3 = \ldots$

c) Check your linear combination by expanding it out to get the original vector $\vec{v}_3$. Does your matrix result from part b) agree with the graphical result from part a)?

d) Now using the new coordinate axes, **draw in** the arrow representing the vector $\vec{v}_4$ whose new coordinates are $(y_1, y_2) = (2, -2)$ and **label** the tip of $\vec{v}_4$ by its symbol. Then draw in the projection parallelogram associated with the new coordinates for which $\vec{v}_4$ is the main diagonal. Read off its old coordinates (x_1, x_2) from the grid (write them down). Do they agree with the linear combination $y_1 \vec{v}_1 + y_2 \vec{v}_2$?



MAT2705-02/05 515 Test2 Answers

① b) $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{v}_3$

$x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}3\\-5\end{bmatrix}$

note that we are using x_1, x_2 for the unknown coefficients here but in the context of the part a) diagram, these are the new coordinates y_1, y_2

$\underbrace{\begin{bmatrix}1 & 3\\1 & -1\end{bmatrix}}_{\substack{A \\ \parallel \\ \langle \vec{v}_1 | \vec{v}_2 \rangle}} \underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{\vec{x}} = \begin{bmatrix}3\\-5\end{bmatrix}$

$A^{-1} = \frac{1}{-1-3}\begin{bmatrix}-1 & -3\\-1 & 1\end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix}1 & 3\\1 & -1\end{bmatrix}$ (or use Maple)

$\vec{x} = A^{-1}\vec{v}_3 = \frac{1}{4}\begin{bmatrix}1 & 3\\1 & -1\end{bmatrix}\begin{bmatrix}3\\-5\end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix}1(3)+3(-5)\\1(3)-1(-5)\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-12\\8\end{bmatrix} = \begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

so $\boxed{\vec{v}_3 = -3\vec{v}_1 + 2\vec{v}_2}$ or explicitly:

$\begin{bmatrix}3\\-5\end{bmatrix} = -3\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}3\\-1\end{bmatrix}$ or

$\langle 3,-5\rangle = \underbrace{-3\langle 1,1\rangle + 2\langle 3,-1\rangle}$

c)

$= \langle -3,-3\rangle + \langle 6,-2\rangle$

$= \langle 3,-5\rangle$ ✓

d) $\vec{v}_4 = 2\vec{v}_1 - 2\vec{v}_2 = 2\begin{bmatrix}1\\1\end{bmatrix} - 2\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}2-6\\2+2\end{bmatrix} = \begin{bmatrix}-4\\4\end{bmatrix}$

yes, these are the old coordinates of $\vec{v}_4$ revealed by the graph. (see graph annotation)

② a) $\underbrace{\begin{bmatrix}3 & 1 & 1 & 6\\1 & -2 & 5 & -5\\4 & 1 & 2 & 7\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix}14\\-7\\17\end{bmatrix}}_{\vec{b}}$

b) $\langle A|\vec{b}\rangle = \begin{bmatrix}3 & 1 & 1 & 6 & 14\\1 & -2 & 5 & -5 & -7\\4 & 1 & 2 & 7 & 17\end{bmatrix}$

$\xrightarrow[\text{Maple}]{\text{RRED}}$ $\begin{array}{c} \begin{matrix} L & L & F & F & \\ x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \begin{bmatrix}1 & 0 & 1 & 1 & 3\\0 & 1 & -2 & 3 & 5\\0 & 0 & 0 & 0 & 0\end{bmatrix}\end{array}$

$x_1 + x_3 + x_4 = 3$
$x_2 - 2x_3 + 3x_4 = 5$
$0 = 0$

$x_3 = t_1$, $x_4 = t_2$

$x_1 = 3 - t_1 - t_2$, $x_2 = 5 + 2t_1 - 3t_2$

② b) continued

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}3-t_1-t_2\\5+2t_1-3t_2\\t_1\\t_2\end{bmatrix}$

c)

$= \underbrace{\begin{bmatrix}3\\5\\0\\0\end{bmatrix}}_{\vec{x}_{part}} + \underbrace{\begin{bmatrix}-t_1-t_2\\2t_1-3t_2\\t_1\\t_2\end{bmatrix}}_{\vec{x}_{hom}}$

OR

$= \underbrace{\begin{bmatrix}3\\5\\0\\0\end{bmatrix}}_{x_{part}} + \underbrace{t_1\begin{bmatrix}-1\\2\\1\\0\end{bmatrix} + t_2\begin{bmatrix}-1\\-3\\0\\1\end{bmatrix}}_{x_{hom}} \equiv t_1\vec{u}_1 + t_2\vec{u}_2$

d) $\vec{b} = A\,\vec{x}_{part} = \boxed{3\vec{v}_1 + 5\vec{v}_2}$.

e) $\{\vec{u}_1, \vec{u}_2\} = \boxed{\left\{\begin{bmatrix}-1\\2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\-3\\0\\1\end{bmatrix}\right\}}$ is a basis of the homogeneous soln space

f) Each of these vectors are the coefficients of an independent linear relationship among the columns of A:

$\vec{u}_1$: $\boxed{-\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}}$
$\vec{u}_2$: $\boxed{-\vec{v}_1 - 3\vec{v}_2 + \vec{v}_4 = \vec{0}}$

③ a) $A = \begin{bmatrix}1 & -2 & 3 & 4\\0 & 3 & 2 & 1\\2 & -3 & 4 & 0\\3 & -2 & 9 & 5\end{bmatrix}$

$|A| = 0$ (Maple) so the columns are $\boxed{\text{linearly dependent}}$

b) $\text{rref}(A) = \begin{array}{c}\begin{bmatrix}1 & 0 & 0 & -71/8\\0 & 1 & 0 & -7/4\\0 & 0 & 1 & 25/8\\0 & 0 & 0 & 0\end{bmatrix} \\ \begin{matrix} L & L & L & F \end{matrix}\end{array}$

$\vec{v}_4$ is an obvious linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ which are a linearly independent subset of the four vectors, so the span of the set of four vectors is a 3-dimensional subspace of $\mathbb{R}^4$.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of that subspace