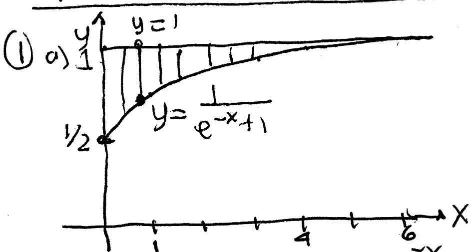
MAT1505-03/04 15F Quiz6

1) a)



$$1-\frac{1}{e^{-x}+1}=\frac{e^{-x}+1-1}{e^{-x}+1}=\frac{e^{-x}}{e^{-x}+1}$$

This explains where the integral comes from. [suggested plot in b)]

$$\int_0^\infty \frac{e^{-x}}{e^{-x}+1}\,dx=\lim_{t\to\infty}\int_0^t \frac{e^{-x}}{e^{-x}+1}\,dx$$

$u=e^{-x}+1 \quad du=-e^{-x}dx$

$x=0:\ u=e^0+1=2$

$x\to\infty:\ u\to$ "$e^{-\infty}+1$" $=\lim_{t\to\infty} e^{-t}+1=1$

$$=\int_2^1 \frac{-du}{u}=\int_1^2\frac{du}{u}=\ln u\Big|_1^2$$

$$=\ln 2-\ln 1=\boxed{\ln 2\approx 0.69}$$

compare

b) $$\int_0^4 \frac{e^{-x}}{e^{-x}+1}\,dx\approx \frac{1}{3}(1)\left(f(0)+4f(1)+2f(2)+4f(3)+f(4)\right)$$

$$\approx 0.67395 \approx \boxed{0.67}$$

$\Delta x=\frac{4}{4}=1$

$f(x)=\frac{e^{-x}}{e^{-x}+1}$

Independent of the Simpson rule approximation we expect the integral for $x=0..4$ to be a bit less than for $x=0..\infty$, but not much since the above plot suggests the exponential decay doesnt contribute much in comparison after $x=4$.

2) a) $$E_{th}=\frac{V\hbar}{\pi^2c^3}\int_0^\infty \frac{\omega^3}{e^{\frac{\hbar\omega}{kT}}-1}\,d\omega=K\int_0^\infty \frac{u^3du}{e^u-1}$$

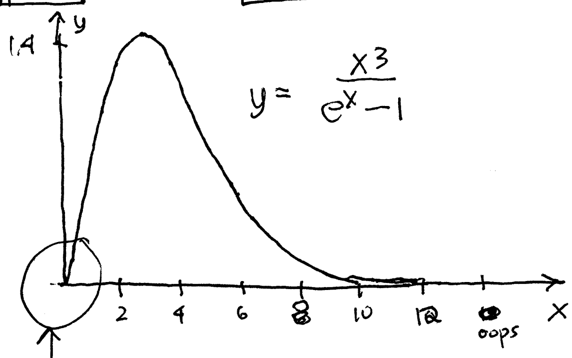
so $u=\frac{\hbar\omega}{kT}$, $du=\frac{\hbar}{kT}d\omega$, $d\omega=\frac{kT}{\hbar}du$, $\omega=\frac{kT}{\hbar}u$

$\omega=0:\ u=0$, $\omega\to\infty:\ u\to\infty$

$$E_{th}=\frac{V\hbar}{\pi^2c^3}\int_0^\infty\frac{\left(\frac{kT}{\hbar}u\right)^3\left(\frac{kT}{\hbar}du\right)}{e^u-1}$$

$$=\underbrace{\frac{V\hbar}{\pi^2c^3}\left(\frac{kT}{\hbar}\right)^4}_{\boxed{K=\frac{Vk^4T^4}{\pi^2\hbar^3c^3}}}\underbrace{\int_0^\infty\frac{u^3du}{e^u-1}}_{\text{b) }=\boxed{\frac{\pi^4}{15}}}\quad \text{c) }=\boxed{\frac{\pi^2Vk^4T^4}{15\hbar^3c^3}}$$

d)



graphically it appears that

$$\lim_{x\to0^+}\frac{x^3}{e^x-1}=0$$

so although the integrand appears to have division by zero at $x=0$, the limit is well-behaved so is no cause for alarm. This is improper only because of the ∞ upper limit.

[in fact by l'Hopital's rule iterated:

$$\lim_{x\to0^+}\frac{x^3}{e^x-1}=\lim_{x\to0^+}\frac{3x^2}{e^x}=\lim_{x\to0^+}\frac{6x}{e^x}=\lim_{x\to0^+}\frac{6}{e^x}=0$$

$\overset{\sim}{\infty/\infty}$ $\overset{\sim}{\infty/\infty}$ $\overset{\sim}{\infty/\infty}$ $\overset{\sim}{\infty/\infty}$]