MAT1505-03/04 15F Quiz 6 Print Name (Last, First) ____________________ , _________________|____

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

1. a) The area between the logistic curve $y = \frac{1}{e^{-x}+1}$ and its horizontal asymptote $y = 1$ for nonnegative x is $\int_0^\infty \frac{e^{-x}}{e^{-x}+1}\, dx$. Convert this to a logarithmic integral by a u-substitution (denominator!) and carefully evaluate this integral using limits. Give its exact and approximate value to 2 decimal places.

b) Use Simpson's rule with 4 divisions to approximate $\int_0^4 \frac{e^{-x}}{e^{-x}+1}\, dx$ to 2 decimal places. Show the formula you evaluate to get this result. Does this seem reasonable compared to part a)? [It wouldn't hurt to sketch the logistic curve and its asymptote for $x = 0..6$ and shade in this area.]

2. The total energy in a cavity of volume V at temperature T of a Planck black body radiation gas (like the cosmic microwave background!) is given by

$$E_{thermal} = \frac{V\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3}{e^{\frac{\hbar\omega}{kT}} - 1}\, d\omega$$

where c is the speed of light, $\hbar$ is Planck's constant, T is the temperature of the radiation, k is Boltzman's constant, and ω is the frequency of the radiation. This is an integral over all frequencies of the energy per unit frequency using the Bose-Einstein distribution function.

a) Show that an obvious u-substitution converts this formula to

$$E_{thermal} = K \int_0^\infty \frac{u^3}{e^u - 1}\, du$$. What is the value of K?

b) Use technology to evaluate the dimensionless definite integral factor. What is its value, exactly and numerically to 2 decimal places?

c) Simplify the final formula for the energy (exactly). [The power law temperature dependence is a key feature of our understanding of the early universe.]

d) Plot the dimensionless integrand $\frac{x^3}{e^x - 1}$ for $x = 0..12$. What is the value of $\lim_{x \to 0^+} \frac{x^3}{e^x - 1}$? How does that make the endpoint 0 not "improper" in the context of this semi-infinite interval integral?

▶ solution[use reverse side or extra paper from bob]