MAT 1505-03/04 15F Final Exam Answers (2)

② a) $A_3 = \overset{\approx 28.3}{9\pi}$, $A_5 = \overset{\approx 78.5}{25\pi}$, $A_4 = \overset{\approx 50.3}{16\pi}$

$\frac{1}{2}(A_3 + A_5) = 17\pi \approx 53.4$

b) $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{81}{(5-4\cos\theta)^2} \, d\theta$ ≈ 47.1

(maple) $= \boxed{15\pi}$ ← closest to area of circle of average radius (16π)

c) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{d}{d\theta}\left(\frac{9\sin\theta}{5-4\cos\theta}\right)}{\frac{d}{d\theta}\left(\frac{9\cos\theta}{5-4\cos\theta}\right)}$

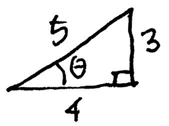
$= \cdots =$ $\boxed{\frac{4-5\cos\theta}{5\sin\theta}}$ $= 0$

Maple + "simplify"

$4 - 5\cos\theta = 0 \to \cos\theta = \frac{4}{5}$, $\boxed{\begin{array}{l}\theta = \arccos\frac{4}{5} \\ \approx 36.9^\circ\end{array}}$

$r = \frac{9}{5-4\cos\theta} = \frac{9}{5-4\left(\frac{4}{5}\right)}$

$= \frac{9}{\frac{25-16}{5}} = \boxed{5}$



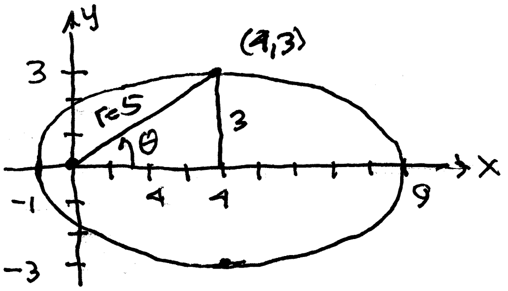
$\sin\theta = 3/5$

$\cos\theta = 4/5$

$x = r\cos\theta = 5\left(\frac{4}{5}\right) = \boxed{4}$

$y = r\sin\theta = 5\left(\frac{3}{5}\right) = \boxed{3}$

(4,3) is directly above the center (4,0) of course!



3-4-5 triangle!

$\theta = \arccos\frac{4}{5} = \arctan\frac{3}{4}$

$\approx 37^\circ$ looks right!

③ a) $x = a\cos t \quad x' = -a\sin t$

$y = b\sin t \quad y' = b\cos t$

$x'^2 + y'^2 = a^2 \underbrace{\sin^2 t}_{1-\cos^2 t} + b^2\cos^2 t$

$= a^2 + \underset{\substack{\downarrow \\ a^2(1-e^2)}}{(b^2 - a^2)}\cos^2 t$

$= a^2 + a^2(1-e^2-1)\cos^2 t$

$= a^2(1 - e^2\cos^2 t)$

$C = \int_0^{2\pi} \sqrt{x'^2 + y'^2}\, dt = a\int_0^{2\pi} \underbrace{\sqrt{1 - e^2\cos^2 t}}\, dt$ ✓

b) $(1 - e^2\cos^2 t)^{1/2} = 1 + \frac{1}{2}(-e^2\cos^2 t) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}(-e^2\cos^2 t)^2 + \cdots$

$= 1 - \frac{1}{2}e^2\cos^2 t - \frac{1}{8}e^4\cos^4 t + \cdots$ (confirmed by Maple "taylor"!!)

Maple

$\doteq 2\pi a - \frac{\pi a}{2}e^2 - \frac{3\pi a}{32}e^4 + \cdots < 2\pi a$ begins decreasing since circle is vertically compressed so shorter circumference.

c) $a = \pi^{-1/2}(1-e^2)^{-1/4}$

$= \pi^{-1/2}\left(1 - \frac{1}{4}(-e^2) - \frac{\frac{1}{4}\left(-\frac{5}{4}\right)}{1\cdot 2}(-e^2)^2 + \cdots\right)$

$= \pi^{-1/2}\left(1 + \frac{1}{4}e^2 + \frac{5}{32}e^4 + \cdots\right)$

$C = 2\pi a\left(1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 + \cdots\right)$

$= 2\pi\pi^{-1/2}\left(1 + \frac{e^2}{4} + \frac{5e^4}{32} + \cdots\right)\left(1 - \frac{e^2}{4} - \frac{3}{64}e^4 + \cdots\right)$

$= \underbrace{2\pi^{1/2}}_{\text{circumference of circle with unit area}}\left(1 + 0e^2 + \frac{3}{64}e^4 + \cdots\right)$ $> 2\pi^{1/2}$

circumference increases with eccentricity, at least for small ϵ.

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① a) $x = 2\cos t - \cos 2t,\ y = 2\sin t - \sin 2t$

$x' = -2\sin t + 2\sin 2t\ ,\ y' = 2\cos t - 2\cos 2t$

$x'^2 + y'^2 = 4(-\sin t + \sin 2t)^2 + 4(\cos t - \cos 2t)^2$

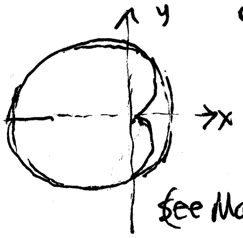
$= 4\left[\sin^2 t + \sin^2 2t - 2\sin t\sin 2t + \underbrace{\cos^2 t}_{4} + \underbrace{\cos^2 2t}_{1} - 2\cos t\cos 2t\right]$

$= 8(1 - \underbrace{(\sin t \sin 2t + \cos t \cos 2t}_{\cos(2t-t) = \cos t}))$ Maple does this simplification automatically.

$= 8(1 - \cos t)$

$\boxed{L = \int_0^{2\pi} \sqrt{8(1-\cos t)}\, dt = 16}$ Maple evaluation

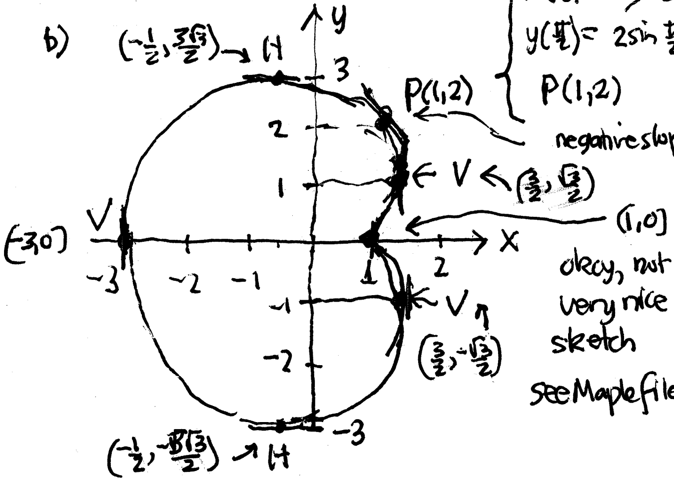
$r = 2.5 \rightarrow C = 2\pi r = 2\pi(2.5) = 5\pi \approx 15.708$



circle closely matches "cardioid" like curve except at its "point" where it is clearly a bit longer so the numbers confirm this.

see Maple plot)

b)



$x(\frac{\pi}{2}) = 2\cancel{\cos\frac{\pi}{2}}^{\,0} - \cos\pi = 1$

$y(\frac{\pi}{2}) = 2\sin\frac{\pi}{2} - \cancel{\sin\pi}_{\,0} = 2$

P(1,2)

negative slope

okay, not very nice sketch

see Maple file

c) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{d}{dt}(2\sin t - \sin 2t)}{\frac{d}{dt}(2\cos t - \cos 2t)} =$

$= \frac{2\cos t - 2\cos 2t}{-2\sin t + 2\sin 2t} = \frac{\cos t - \cos 2t}{-\sin t + \sin 2t}$

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{\cancel{\cos\frac{\pi}{2}}^{0} - \cos\pi}{-\sin\frac{\pi}{2} + \cancel{\sin\pi}_{0}} = \frac{-(-1)}{-1} = \boxed{-1}$

looks perfect compared to my Maple plot.

$y - 2 = -1(x-1) \rightarrow y = 2 - (x-1)$

$= 2 + 1 - x = 3 - x$

$\boxed{y = 3 - x}$

① d) $\frac{dy}{dx} = \frac{\cos t - \overset{2\cos^2 t - 1}{\overbrace{\cos 2t}}}{-\sin t + \underbrace{\sin 2t}_{2\sin t\cos t}}$ $\rightarrow = 0$ H tanline ? / $\rightarrow = 0$ V tanline ?

$\underbrace{\sin t}_{=0}(\underbrace{-1 + 2\cos t}_{=0}) = 0$

$t = 0, \pi$ $\cos t = +\frac{1}{2}$

$t = \pm\frac{\pi}{3}$

Maple gives all of these

$2\cos^2 t - \cos t - 1 = 0$

$\cos t = \frac{1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)}$

$= \frac{1\pm 3}{4} = 1, -\frac{1}{2}$

$t = 0$ $t = \pm\frac{2\pi}{3}$

Maple misses obvious reflected point $t = -\frac{2\pi}{3}$

$t = 0$

$\frac{dy}{dx} \sim \frac{0}{0} \longrightarrow$ l'hopital,

$\lim_{t\to 0}\frac{dy}{dx} = \lim_{t\to 0}\frac{\cos t - \cos 2t}{-\sin t + \sin 2t} = \lim_{t\to 0}\frac{-\sin t + 2\sin 2t}{-\cos t + 2\cos 2t}$

$= \frac{-0 + 0}{-1 + 2} = \frac{0}{1} = 0$

so limiting tangent line is horizontal

H:

$t = 0$: $x = 2 - 1 = 1$, $y = 0 - 0 = 0$ (1,0)

$t = \pm\frac{2\pi}{3}$: $x = 2\cos\frac{2\pi}{3} - \cos\frac{4\pi}{3} = 2(-\frac{1}{2}) - (-\frac{1}{2}) = -\frac{1}{2}$

$y = \pm(2\sin\frac{2\pi}{3} - \sin\frac{4\pi}{3}) = \pm[2(\frac{\sqrt3}{2}) - (-\frac{\sqrt3}{2})] = \pm\frac{3\sqrt3}{2}$

e) $(-\frac{1}{2}, \pm\frac{3\sqrt3}{2}) \approx (-0.5, 2.60)$

V: $t = \pm\frac{\pi}{3}$; $x = 2\cos(\frac{\pi}{3}) - \cos(\frac{2\pi}{3}) = 2(\frac{1}{2}) - (-\frac{1}{2}) = \frac{3}{2}$

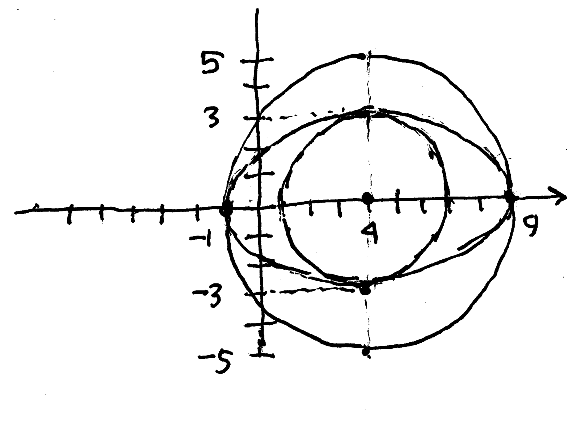
$y = \pm(2\sin\frac{\pi}{3} - \sin\frac{2\pi}{3}) = \pm(2(\frac{\sqrt3}{2}) - \frac{\sqrt3}{2}) = \pm\frac{\sqrt3}{2}$

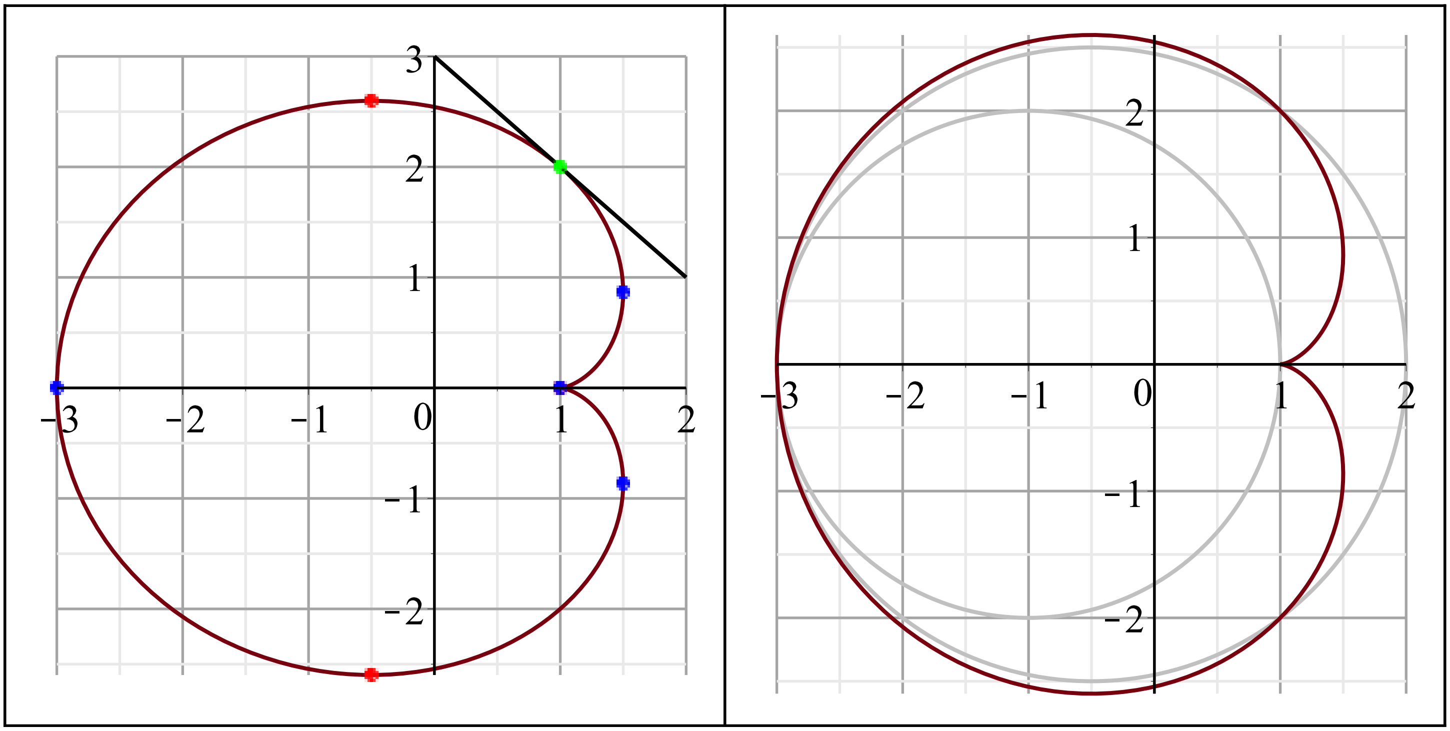
e) $(\frac{3}{2}, \pm\frac{\sqrt3}{2}) \approx (1.5, 0.867)$

$t = \pi$: $x = 2\cos\pi - \cos(2\pi) = -2 - 1 = -3$, $y = 2\sin\pi - \sin(2\pi) = 0$ (-3,0)

all these points agree with Maple plot gridlines

②





3
2
1
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0
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2
−1
−2