MAT1505-03/04 15F Final Exam   Print Name (Last, First) ________________________________|____

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

1. For the parametrized curve: $x = 2\cos(t) - \cos(2t),\ y = 2\sin(t) - \sin(2t),\ t = 0..2\pi$:

a) Write down an integral formula for the arclength of this curve, simplifying the sum of squares inside the square root integrand with Maple, then evaluate the integral exactly with Maple. [The simple exact result follows from a perfect square simplification using a half-angle identity, after using the cosine difference formula to simplify it.] How does its numerical value compare to the circumference of the comparison circle of radius 2.5 centered at $\left(-\frac{1}{2}, 0\right)$, namely $x = -\frac{1}{2} + 2.5\cos(t),\ y = 2.5\sin(t),\ t = 0..2\pi$? Explain.

b) Using a technology plot with equal tickmarks on both axes, make a hand sketch of this curve and mark off the points where the tangent line is horizontal (H), or vertical (V) by putting a bullet mark at those points and labeling them as H or V. Also mark off the point P on the curve where $t = \frac{\pi}{2}$. Draw in a piece of the tangent line at P. [Label your tickmarks and axes!]

c) Evaluate the slope $\frac{dy}{dx}$ at the point P. Does this value seem to agree with your sketch? Write an equation for this tangent line.

d) Now find the exact coordinates (x, y) of the points labeled H and V, using Maple to solve the necessary conditions, using symmetry to add any points Maple misses, and annotate your sketch with the (x, y) values. [Hint: the t parameter values are all multiples of $\frac{\pi}{3}$.] What happens to the limiting slope at $t = 0$?

e) Evaluate the coordinates of these points numerically and verify that they correspond to the points you marked off in part b).

2. The parametrized curve $x = 4 + 5\cos(t),\ y = 3\sin(t)$ traces out the polar coordinate graph $r = \frac{9}{5 - 4\cos(\theta)}$, which is an ellipse centered at $(4, 0)$.

a) Make a rough sketch of your technology plot of this curve. Evaluate the area of the inscribed $(r = 3)$, circumscribed $(r = 5)$ circles, and their average area, as well as the area of the circle with the average radius $(r = 4)$, and give the corresponding numerical values.

b) Use the polar coordinate area formula to evaluate the area of the ellipse exactly. Is its value closer to the average circle area or the area of the circle of average radius?

c) Write down a generic formula the slope dy/dx of the tangent line in polar coordinates, and use Maple to evaluate and simplify that formula for this particular polar curve. Write down the result. Use it to find the exact values of the polar coordinates of the upper vertex of the ellipse (where the tangent line is horizontal), and give the one decimal place degree equivalent of the polar angle. Evaluate the corresponding Cartesian coordinates exactly (this can be done easily by hand). Do they agree with their obvious values in the Cartesian parametrized curve representation?

(see next page)

3. We showed in a homework exercise that for an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with semiaxes $a > b > 0$, the area enclosed by the ellipse is $A = \pi\, a\, b$. The eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}}$ measures the deviation of the ellipse from a circle, corresponding to $e = 0$.

The minor semiaxis b expressed in terms of the major semiaxis a and the eccentricity e is $b = a\sqrt{1 - e^2}$. If we look at the family of ellipses with unit area then

$1 = \pi\, a\, b = \pi\, a^2 \sqrt{1 - e^2}$ so that solving for a we find $a = \pi^{-\frac{1}{2}} \left(1 - e^2\right)^{-\frac{1}{4}}$, the limit $e = 0$ corresponds to unit area circle radius $a = \pi^{-\frac{1}{2}}$ with circumference $C = 2\,\pi\,\pi^{-\frac{1}{2}} = 2\,\pi^{\frac{1}{2}}$.

a) These ellipses can be parametrized by

$x = a\cos(t), y = b\sin(t), t = 0 .. 2\,\pi$ where $b = a\sqrt{1 - e^2}$ and $a > b > 0$.

Show that their circumferences can be written

$$C = a\int_0^{2\pi} \sqrt{1 - e^2 \cos^2(t)}\ \mathrm{d}t$$

b) Use the binomial series Taylor expansion of the integrand about $e = 0$ to write out the first 3 nonzero terms of the integrand, then integrate term by term (use Maple). This shows how the circumference begins to change as we increase the eccentricity.

c) **Optional.**

Now expand the relation $a = \pi^{-\frac{1}{2}} \left(1 - e^2\right)^{-\frac{1}{4}}$ in a similar binomial Taylor expansion up to the first 3 nonzero terms, and multiply out the product expression for the circumference C, neglecting terms higher than e^4.

Obtain the result: $C(e) = C(0)\left(1 + k\, e^4\right)$, where $k > 0$. What is the value of k?

This shows that within this family of curves enclosing unit area, increasingly deforming the circle (with circumference $C(0)$) initially leads to increasing the circumference from its minimal value. The circle in fact is the minimal perimeter curve enclosing a fixed area in the plane, and this calculation confirms this fact within this family of curves.

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet in on top of your answer sheets as a cover page, with the first test page facing up:

"During this examination, all work has been my own. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: