MAT1505-03/04 15F Takehome Test 3   Print Name (Last, First) ______________________________|____
Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

1. Which of the following series converge (and explain why or why not)?

a) $\sum_{n=1}^{\infty} \frac{e^{1-\frac{1}{n}}}{2^n}$, b) $\sum_{n=1}^{\infty} \frac{3^{n+1}}{n^2\, 2^{2n}}$, c) $\sum_{n=1}^{\infty} \frac{\pi}{n} \sin\left(\frac{1}{n}\right)$, d) $\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$

2. Use a Taylor series expansion to approximate $\sin(95°)$ with an error of less than 0.0001.

3. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n\, 3\,(x-2)^n}{2^{2n}(n+1)}$, carefully justifying your claims.

4. Find the Taylor series for $f(x) = x\, e^x$ and integrating the result term by term, using the fact that $\int_0^1 x\, e^x\, dx = 1$, show that $\sum_{n=0}^{\infty} \frac{1}{(n+2)\, n!} = 1$.

5. a) Use technology to verify the integral formula: $\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x) + C$.
b) Use this and the binomial series to evaluate the first three (nonzero) terms of the Taylor series for arcsin.
c) Now verify your result by evaluating the Taylor series formula for the first three (nonzero) terms of arcsin (use technology to obtain the simplified derivative functions and state them as well as their values at $x = 0$). Does this agree with Maple's taylor command?

6. a) Use the binomial series to write out the first 5 terms of the power series expansion of $f(x) = (1+x)^{-2}$ and deduce from the obvious pattern a formula for the nth term in the series, i.e., express the result as an explicit formula $f(x) = \sum_{n=0}^{\infty} a_n x^n$.
b) Check that your formula is correct by using the derivative of the series $(1+x)^{-1}$, itself obtained from the formula for the sum of a geometric series with ratio $-x$.
c) Let r be the Earth-Moon distance between their centers and let R be the radius of the Earth and let G be Newton's gravitational constant. The tidal force of gravity exerted on the Earth by the Moon (part of explaining the ocean tides) is given by the formula $F_{tidal} = \frac{G M m}{(r-R)^2} - \frac{G M m}{(r+R)^2}$. Expand this force in a power series in the small ratio $\frac{R}{r}$, and calculate the leading two terms.
d) Express the result as $F_{tidal} = a\,(1+b)$ so that b is the fractional error in using a to describe this tidal effect (neglecting the rest of the series). Evaluate a and b for the average distance $r = 384,400$ km, average Earth radius $R = 6371$ km, the lunar mass is $m = 0.07342\ 10^{24}$kg, the Earth mass is $M = 5.9726\ 10^{24}$kg, and $G = 6.6726\ 10^{-11}$ in MKS units. How many significant figures should we keep in these two evaluations?

Optional fun (translation: ignore this).
You can Google how the tangent addition formula is used to derive the amazing identity $\pi = 4\left(4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)\right)$. Using the Taylor series for arctan and Maple, how many (nonzero) terms in this series are needed to get the approximation 3.14159, which is the most digits dr bob's brain will hold long term?

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet in on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: