MAT1505-03/04 15F Test2 Answers

① $f(x)=\int_0^x \sqrt{4\cos^2 t-1}\,dt = y$

$\frac{dy}{dx}=\sqrt{4\cos^2 x-1}$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+(4\cos^2 x-1)}=\sqrt{4\cos^2 x}$

$=2|\cos x| = 2\cos x$ on $[-\frac{\pi}{3},\frac{\pi}{3}]$

$L=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos x\,dx = 2\sin x\Big|_{-\pi/3}^{\pi/3}$

$=4\sin\frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right)=\boxed{2\sqrt{3} \approx 3.4641}$

straight line between approx endpts: $\pm(1,1.3)$

$d=\sqrt{(1-(-1))^2+(1.3-(-1.3))^2}=\sqrt{2^2+2.6^2}$

≈ 3.28 curve is a bit longer as it should be ✓

② $\int_2^4 \frac{y}{\sqrt{y-2}}\,dy = \lim_{a\to 2^+}\int_a^4 \frac{y}{\sqrt{y-2}}\,dy$

$=\lim_{a\to 2^+} \frac{2}{3}\sqrt{y-2}\,(y+4)\Big|_a^4$

$=\lim_{a\to 2^+}\left(\frac{2}{3}\sqrt{2}(8)-\frac{2}{3}\underbrace{\sqrt{a-2}}_{\to 0}(a+4)\right)$

$=\boxed{\frac{16\sqrt{2}}{3}} \approx 7.5425$

③ a) $\int_0^\infty P(x)\,dx=\int_0^\infty \frac{x}{a^2}e^{-\frac{x}{a}}\,dx$

$=\lim_{t\to\infty}\int_0^t \frac{x}{a^2}e^{-\frac{x}{a}}\,dx$

$=\lim_{t\to\infty} -\frac{(x+a)}{a}e^{-\frac{x}{a}}\Big|_0^t$

$=\lim_{t\to\infty} -\frac{t+a}{a}e^{-\frac{t}{a}}+1e^0$

$=1-\underbrace{\lim_{t\to\infty}\frac{t+a}{a e^{t/a}}}\approx 1$ ✓

$=\lim_{t\to\infty}\frac{1}{a\frac{1}{a}e^{t/a}}=0$

l'Hopital's rule for $\frac{\infty}{\infty}$ limit

③ b) $\langle x\rangle=\int_0^\infty x\,P(x)\,dx=\int_0^\infty \frac{x^2}{a^2}e^{-\frac{x}{a}}\,dx$

$=\lim_{t\to\infty}\int_0^t \frac{x^2}{a^2}e^{-\frac{x}{a}}\,dx$

$=\lim_{t\to\infty} -\frac{(2a^2+2ax+x^2)}{a}e^{-\frac{x}{a}}\Big|_0^t$

$=\lim_{t\to\infty}\left(\frac{2a^2+0}{a}e^0-\underbrace{\frac{(2a^2+2at+t^2)}{a}e^{-\frac{t}{a}}}\right)$

$\frac{\infty}{\infty}$ limit $\to 0$ by 2 applications of L'Hopital's rule

$=\boxed{2a}$

c) $P'(x)=\frac{d}{dx}\left(\frac{x}{a^2}e^{-\frac{x}{a}}\right)=\frac{1}{a^2}e^{-\frac{x}{a}}+\frac{x}{a^2}\left(-\frac{1}{a}\right)e^{-\frac{x}{a}}$

$=\frac{1}{a^3}(a-x)\,e^{-\frac{x}{a}}=0 \quad\to x=a.$

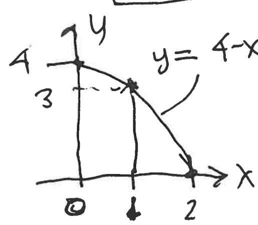
$\boxed{x_{peak}=a}$

d) $P(x<x_{peak})=\int_0^{x_{peak}}\frac{x}{a^2}e^{-\frac{x}{a}}\,dx$

$=-\frac{(x+a)}{a}e^{-\frac{x}{a}}\Big|_0^a=-\left(\frac{a+a}{a}\right)e^{-\frac{a}{a}}+\frac{a}{a}e^0$

$=\boxed{1-2e^{-1}\approx 0.2642}$

④



$y=4-x^2=f(x)$

$\Delta x=1$

$\int_0^2 4-x^2\,dx\approx\frac{1}{3}(1)\left(f(0)+4f(1)+f(2)\right)$

$=\frac{1}{3}(4+4(3))=\boxed{\frac{16}{3}}=A_{simpson,2}$

$=4x-\frac{x^3}{3}\Big|_0^2=8-\frac{8}{3}=\frac{16}{3}$ ✓ (Simpson's rule is exact for quadratic functions)

If one takes the exact length of the secant line connecting the endpoints, one finds

$d=\sqrt{\left(\frac{\pi}{3}-\left(-\frac{\pi}{3}\right)\right)^2+\left(f\left(\frac{\pi}{3}\right)-f\left(-\frac{\pi}{3}\right)\right)^2}\approx 3.41$

which is even closer.

If you found length of $y=\sqrt{4\cos^2 x-1}$ you find instead

$L\approx 4.33$ which is $\frac{4.33}{3.41}\sim 1.27$

over 1/4 longer, too long, unreasonable