MAT2500-03/04 14S Test 1 Answers

a) $\vec{r}(t) = \langle 2e^t, e^{2t}, \frac{1}{3}e^{3t}\rangle = \langle x,y,z\rangle$

$z=\frac{1}{3}e^{3t}$ is an increasing function of t so the curve moves up

$$\boxed{\begin{aligned}\vec{r}\,'(t) &= \langle 2e^t, 2e^{2t}, e^{3t}\rangle\\ \vec{r}\,''(t) &= \langle 2e^t, 4e^{2t}, 3e^{3t}\rangle\end{aligned}}$$

$$|\vec{r}\,'(t)| = \sqrt{4e^{2t}+4e^{4t}+e^{6t}} = \sqrt{e^{2t}(4+4e^{2t}+e^{4t})} = e^t\sqrt{(2+e^{2t})^2} = \boxed{e^t(2+e^{2t})} \text{ (factored)}$$

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 2e^t, 2e^{2t}, e^{3t}\rangle}{e^t(2+e^{2t})} = \boxed{\frac{\langle 2, 2e^t, e^{2t}\rangle}{2+e^{2t}}}$$

$$\boxed{\begin{aligned}|\vec{r}\,''(t)| &= \sqrt{4e^{2t}+16e^{4t}+9e^{6t}}\\ &= e^t\sqrt{4+16e^{2t}+9e^{4t}}\end{aligned}} \text{ (factored)}$$

$$\boxed{\begin{aligned}&\vec{r}(0) = \langle 2,1,\tfrac{1}{3}\rangle & &\hat{T}(0) = \tfrac{1}{3}\langle 2,2,1\rangle\\ &\vec{r}\,'(0) = \langle 2,2,1\rangle & &|\vec{r}\,'(0)| = \sqrt{4+4+1} = \sqrt{9} = 3\\ &\vec{r}\,''(0) = \underset{=\vec{a}(0)}{\langle 2,4,3\rangle} & &|\vec{r}\,''(0)| = \sqrt{4+16+9} = \sqrt{29}\end{aligned}}$$

b) $\vec{r}_0 = \vec{r}(0) = \langle 2,1,1/3\rangle$

$\vec{a} = \vec{r}\,'(0) = \langle 2,2,1\rangle$

$\vec{r} = \vec{r}_0 + t\vec{a} = \langle 2,1,1/3\rangle + t\langle 2,2,1\rangle$

$= \boxed{\langle 2+2t, 1+2t, 1/3+t\rangle = \langle x,y,z\rangle}$

or

$\boxed{x = 2+2t,\ y = 1+2t,\ z = 1/3+t}$

c) $\vec{b}(t) = \vec{r}\,'(t)\times\vec{r}\,''(t) = \langle 2e^t, 2e^{2t}, e^{3t}\rangle \times \langle 2e^t, 4e^{2t}, 3e^{3t}\rangle$

$$= \begin{vmatrix} i & j & k\\ 2e^t & 2e^{2t} & e^{3t}\\ 2e^t & 4e^{2t} & 3e^{3t}\end{vmatrix} = \langle 6e^{5t}-4e^{5t}, 2e^{4t}-6e^{4t}, 8e^{3t}-4e^{3t}\rangle$$

$= \boxed{\langle 2e^{5t}, -4e^{4t}, 4e^{3t}\rangle}$ or use Maple

$= \boxed{2e^{3t}\langle e^{2t}, -2e^t, 2\rangle}$ (factored)

$$|\vec{b}(t)| = 2e^{3t}\sqrt{\underbrace{e^{4t}+4e^{2t}+4}_{(e^{2t}+2)^2}} = 2e^{3t}(2+e^{2t})$$

$\boxed{\vec{b}(0) = \langle 2,-4,4\rangle} = 2\langle 1,-2,2\rangle$

$|\vec{b}(0)| = 2\cdot 1\cdot(2+1) = \boxed{6}$

d) choose normal $\vec{n} = \langle 1,-2,2\rangle$, $\vec{r}_0 = \langle 2,1,1/3\rangle$

$0 = \vec{n}\cdot(\vec{r}-\vec{r}_0) = \langle 1,-2,2\rangle\cdot\langle x-2, y-1, z-1/3\rangle$

$= x-2-2(y-1)+2(z-1/3) = x-2y+2z\cancel{-2+2}-2/3$

so $\boxed{x-2y+2z = 2/3}$

e) $$K(t) = \frac{|\vec{b}(t)|}{|\vec{r}\,'(t)|^3} = \frac{2e^{3t}(2+e^{2t})}{(e^t(2+e^{2t}))^3} = \boxed{\frac{2}{(2+e^{2t})^2}}$$

$$\boxed{\rho(t) = \frac{(2+e^{2t})^2}{2}} \qquad \rho(0) = \frac{(2+1)^2}{2} = \boxed{\frac{9}{2}}$$

f) $$\vec{B}(t) = \frac{\vec{b}(t)}{|\vec{b}(t)|} = \frac{2e^{3t}\langle e^{2t}, -2e^t, 2\rangle}{2e^{3t}(2+e^{2t})} = \boxed{\frac{\langle e^{2t}, -2e^t, 2\rangle}{2+e^{2t}}}$$

$$\boxed{\vec{B}(0) = \frac{\langle 1,-2,2\rangle}{3}}$$

g) $\vec{N}(0) = \vec{B}(0)\times\hat{T}(0) = \frac{1}{3}\langle 1,-2,2\rangle \times \frac{1}{3}\langle 2,2,1\rangle$

$$= \frac{1}{9}\begin{vmatrix} i & j & k\\ 1 & -2 & 2\\ 2 & 2 & 1\end{vmatrix} = \frac{1}{9}\langle -2-4, 4-1, 2+4\rangle = 1/9\,\langle -6,3,6\rangle = \boxed{\tfrac{1}{3}\langle -2,1,2\rangle}$$

h) $a_T(0) = \hat{T}(0)\cdot\vec{a}(0) = \frac{1}{3}\langle 2,2,1\rangle\cdot\langle 2,4,3\rangle$

$= \frac{1}{3}(4+8+3) = \frac{15}{3} = \boxed{5}$

$a_N(0) = \hat{N}(0)\cdot\vec{a}(0) = \frac{1}{3}\langle -2,1,2\rangle\cdot\langle 2,4,3\rangle$

$= \frac{1}{3}(-4+4+6) = \boxed{2}$

$a_T(0)^2 + a_N(0)^2 = 5^2+2^2 = 29 = |\vec{a}(0)|^2$ ✓

i) $\vec{C}(0) = \vec{r}(0) + \rho(0)\vec{N}(0)$

$= \langle 2,1,1/3\rangle + \underbrace{\frac{9}{2}\cdot\frac{1}{3}}_{3/2}\langle -2,1,2\rangle$

$= \langle 2,1,1/3\rangle + \langle -3, 3/2, 3\rangle$

$= \langle -1, \frac{5}{2}, \frac{10}{3}\rangle$ ✓

j) $L = \int_0^1 |\vec{r}\,'(t)|\,dt$

$= \int_0^1 e^t(2+e^{2t})\,dt$

$= \int_0^1 2e^t + e^{3t}\,dt$

$= 2e^t + \frac{1}{3}e^{3t}\Big|_0^1$

$= 2e + \frac{1}{3}e^3 - (2+1/3)$

$= \boxed{2e + \tfrac{1}{3}e^3 - 7/3}$

$\approx \boxed{9.7984}$