MAT2705-01/02 14F Quiz 10 NOT! Print Name (Last, First) ________________________________ |__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. Given: $\lambda_1 = -\frac{1}{2} + 3\,I$, $\lambda_2 = -\frac{1}{2} - 3\,I$, $\vec{b}_1 = \langle 1 - 3\,I, 1\rangle$, $\vec{b}_2 = \langle 1 + 3\,I, 1\rangle$, $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$

a) Evaluate the real and imaginary parts of $z = e^{\lambda_1 t}$.

b) Evaluate the real and imaginary parts of $\vec{z} = e^{\lambda_1 t}\vec{b}_1 = \vec{x} + I\vec{y}$.

c) Let $\vec{u} = a\,\vec{x} + b\,\vec{y}$. Solve the condition $\vec{u}(0) = \langle 2, -1\rangle$ for (a, b), backsubstitute into $\vec{u}$ and simplify.

d) Express the sinusoidal factor in each vector component of $\vec{u}$ in phase shifted form.

e) Plot u_1 and u_2 versus t (use the original expressions, not the phase-shifted ones) for 5 characteristic times of the exponential factor starting at $t = 0$, including the envelopes of both decaying oscillations.

f) Which of these two functions peaks earlier in time? Explain your reasoning.

g) What fraction of a cycle separates these peaks? How many peaks do you see in the viewing window for each function?

h) Evaluate $A = B \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} B^{-1}$, then use Maple to solve the IVP $\vec{u}' = A\,\vec{u}$, $\vec{u}(0) = \langle 2, -1\rangle$ as a check on your work.

▶ solution

① a) $z = e^{(-\frac{1}{2}+3i)t} = e^{-t/2} e^{3it} = e^{-t/2}(\cos 3t + i \sin 3t)$

$= \underbrace{e^{-t/2}\cos 3t}_{Re(z)} + i\, \underbrace{e^{-t/2}\sin 3t}_{Im(z)}$

b) $\vec{z} = e^{(-\frac{1}{2}+3i)t}\begin{bmatrix} 1-3i \\ 1 \end{bmatrix} = e^{-t/2}(\cos 3t + i \sin 3t)\begin{bmatrix} 1-3i \\ 1 \end{bmatrix}$

$= e^{-t/2}\begin{bmatrix} (\cos 3t + i \sin 3t)(1-3i) \\ \cos 3t + i \sin 3t \end{bmatrix}$

$= e^{-t/2}\begin{bmatrix} \cos 3t + 3\sin 3t + i(\sin 3t - 3\cos 3t) \\ \cos 3t + i \sin 3t \end{bmatrix}$

$= \underbrace{e^{-t/2}\begin{bmatrix} \cos 3t + 3\sin 3t \\ \cos 3t \end{bmatrix}}_{\vec{x} = Re(\vec{z})} + i\, \underbrace{e^{-t/2}\begin{bmatrix} \sin 3t - 3\cos 3t \\ \sin 3t \end{bmatrix}}_{\vec{y} = Im(\vec{z})}$

c) $\vec{u} = a\vec{x} + b\vec{y} = e^{-t/2}\begin{bmatrix} a(\cos 3t + 3\sin 3t) + b(\sin 3t - 3\cos 3t) \\ a\cos 3t + b \sin 3t \end{bmatrix}$

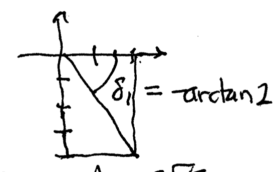
$\vec{u}(0) = \begin{bmatrix} a - 3b \\ a \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad a - 3b = 2 \to b = \frac{1}{3}(a-2) = \frac{1}{3}(-3) = -1$, $\quad a = -1$

$\vec{u} = e^{-t/2}\begin{bmatrix} -(\cos 3t + 3\sin 3t) - (\sin 3t - 3\cos 3t) \\ -\cos 3t - \sin 3t \end{bmatrix}$

$\boxed{\vec{u} = e^{-t/2}\begin{bmatrix} 2\cos 3t - 4\sin 3t \\ -\cos 3t - \sin 3t \end{bmatrix}}$

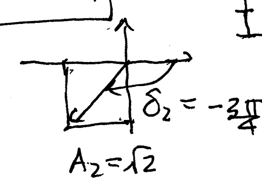
d) $2\cos 3t - 4\sin 3t$:

$= \boxed{2\sqrt{5}\cos(3t + \arctan 2)}$



$-\cos 3t - \sin 3t$

$= \boxed{\sqrt{2}\cos(3t + \frac{3\pi}{4})}$



e) u_1 envelope: $\pm 2\sqrt{5}\, e^{-t/2}$

u_2 envelope: $\pm\sqrt{2}\, e^{-t/2}$

$\tau = 2$, $5\tau = 10$ so $t = 0..10$ (see plot) next page

f) u_2 is shifted more to the left, so $\boxed{\text{earlier}}$ in time compared to u_1 $\quad (\delta_2 < \delta_1 < 0)$

g) $\delta_1 - \delta_2 = \frac{3\pi}{4} - \arctan 2 \approx 71.6°$

$\frac{\delta_1 - \delta_2}{2\pi} \approx 0.199$ cycles $\sim \frac{1}{5}$ period,

h) $B A B^{-1} \overset{\text{maple}}{=} \begin{bmatrix} -3/2 & 10 \\ -1 & 1/2 \end{bmatrix}$

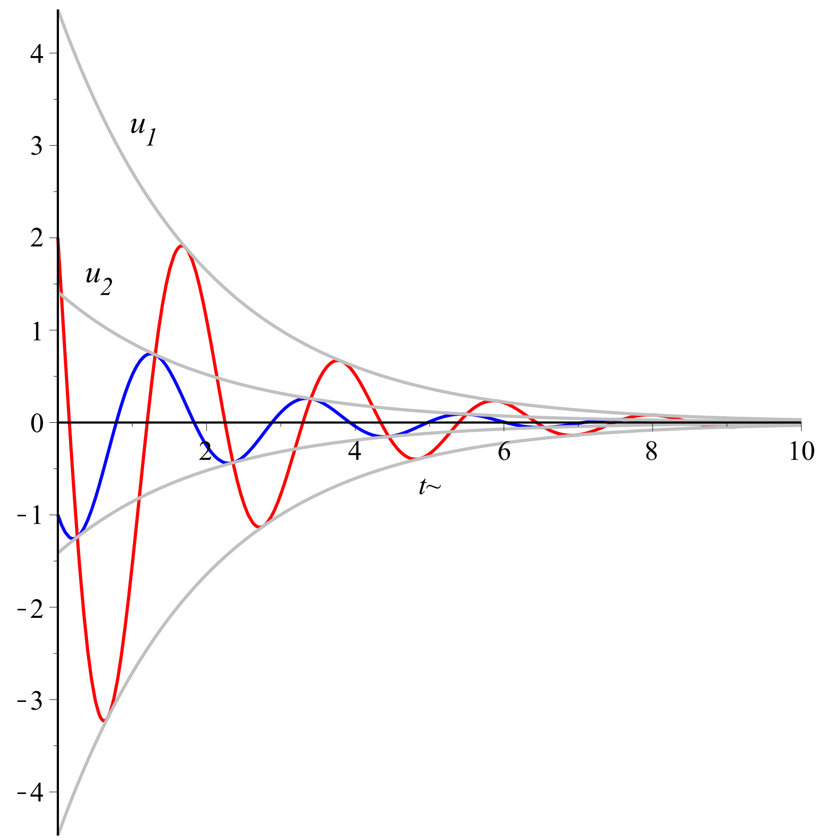
> x1'(t) = -3/2 x1(t) + 10 x2(t),
x2'(t) = - x1(t) + 1/2 x2(t),
x1(0) = 2, x2(0) = -1

rightclick, solve DE system:

$\boxed{\begin{aligned} x1(t) &= e^{-\frac{1}{2}t}(2\cos(3t) - 4\sin(3t)), \\ x2(t) &= e^{-\frac{1}{2}t}(-\cos(3t) - \sin(3t)) \end{aligned}}$ (with simplification)

g) continued: $T = 2\pi/3 \approx 2.09$, $5\tau/T \approx 5$

but from $t = 8..10$ can't see much so 4 cycles are clearly visible



4
3
2
1
0
- 1
- 2
- 3
- 4
2
4
6
8
10
t~
u1
u2