MAT2705-01/02 14F Quiz 9 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). You may use technology to give the RREF form of matrices.

1. $x_1'(t) = -4\,x_1(t) + 2\,x_2(t) - x_3(t)$, $x_2'(t) = -2\,x_1(t) - x_3(t)$, $x_3'(t) = -2\,x_1(t) + 2\,x_2(t) - 3\,x_3(t)$, $x_1(0) = 2, x_2(0) = 1, x_3(0) = -3$.

a) Rewrite this system of DEs **and** its initial conditions explicitly in matrix form for the vector variable $\vec{x} = \langle x_1, x_2, x_3\rangle$ as a column matrix (using the actual matrices, not their symbols), identifying the coefficient matrix A.

b) For this A, using Maple write down the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (ordered by increasing absolute value, they are integers!) and corresponding matrix of eigenvectors $B_{maple} = \langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle$ that it provides you, reordering them if necessary to order them as requested.

c) By hand showing all steps (you should use technology to evaluate the necessary determinant and to solve the resulting condition), show that the characteristic equation (state it explicitly) for the eigenvalues has the same roots as Maple's eigenvalue list. What are their respective multiplicities?

d) For each eigenvalue, by hand find a basis of the corresponding eigenspace, collecting your results into a new matrix B and compare your result with Maple's. Do they agree once reordered as above? If not, are they equivalent modulo permutations or rescalings?

e) Use technology to evaluate and write down the inverse matrix B^{-1} and use Maple to evaluate the matrix product $A_D = B^{-1} A\, B$. Write down this result. Does it evaluate correctly to the diagonalized matrix with the eigenvalues in the correct order?

f) Given that $\vec{x} = B\,\vec{y}$, if $\vec{x}(0) = \langle 2, 1, -3\rangle$, find $\vec{y}(0)$. Show how you did this.

g) Write down Maple's solution of this IVP for $\vec{x}(t)$ and compare it to the linear combination:

$$y_1(0)\, e^{\lambda_1 t}\,\vec{b}_1 + y_2(0)\, e^{\lambda_2 t}\,\vec{b}_2 + y_3(0)\, e^{\lambda_3 t}\,\vec{b}_3.$$

▶ solution

① a) $$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \overbrace{\begin{bmatrix} -4 & 2 & -1 \\ -2 & 0 & -1 \\ -2 & 2 & -3 \end{bmatrix}}^{A} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

function notation optional!

b) $$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}, \quad B_{maple} = \begin{bmatrix} -1/2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

($|-2| < |-3|$ increasing absolute value)

c) $$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 2 & -1 \\ -2 & -\lambda & -1 \\ -2 & 2 & -3-\lambda \end{vmatrix}$$

$$\overset{\text{det}}{=} -\lambda^3 - 7\lambda^2 - 16\lambda - 12$$

$$\overset{\text{factor}}{=} -(\lambda+2)^2(\lambda+3) = 0$$

$$\boxed{\lambda = \underbrace{-2, -2}_{m=2}, \underbrace{-3}_{m=1}}$$

repeated eigenvalue

d) $\lambda = -2$: $$A + 2I = \begin{bmatrix} -2 & 2 & -1 \\ -2 & 2 & -1 \\ -2 & 2 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{ccc} L & F & F \\ \hline \end{array} \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

d) continued

$x_2 = t_1$, $x_3 = t_2$ $x_1 - t_1 + \frac{1}{2} t_2 = 0$, $x_1 = t_1 - \frac{1}{2} t_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t_1 - \frac{1}{2} t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}_1} + t_2 \underbrace{\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{b}_2}$$

$\lambda = -3$: $$A + 3I = \begin{bmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ -2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{array}{ccc} L & L & F \end{array} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = t$, $x_1 = t$, $x_2 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{b}_3}, \quad \boxed{B = \begin{bmatrix} 1 & -1/2 & 1 \\ [illegible] & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$$

Maple interchanges first two eigenvectors since it names free variable parameters from bottom up.

e) $$B^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ -2 & 2 & 0 \\ 2 & -2 & 1 \end{bmatrix} \quad A_D = B^{-1} A B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

yes, diagonal values are eigenvalues in chosen order

f) $$\vec{y}(0) = B^{-1}\vec{x}(0) \overset{\text{Maple}}{=} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \text{or: } \begin{bmatrix} -2 & 3 & -1 \\ -2 & 2 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -4+3+3 \\ -4+2 \\ 4-2-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

g) $$\vec{x}(t) = \begin{bmatrix} 3e^{-2t} - e^{-3t} \\ 2e^{-2t} - e^{-3t} \\ -2e^{-2t} - e^{-3t} \end{bmatrix} = \underbrace{2e^{-2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 2e^{-2t}\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}}_{e^{-2t}\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} \checkmark} + e^{-3t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

e^{-2t} terms: slower decay; e^{-3t} terms: faster decay

they agree!