① a) $4x'' + 4x' + 37x = F$

$$\boxed{x'' + x' + \tfrac{37}{4}x = F/4}$$

$\omega_0^2 = 37/4$

$k_0 = \boxed{1}$

$\tau_0 = 1/k_0 = \boxed{1}$ $\quad \omega_0 = \boxed{\sqrt{37}/2 \approx 3.041}$

$Q = \omega_0\tau_0 = \boxed{\sqrt{37}/2 \approx 3.041} > \frac{1}{2}$ underdamped

$T_0 = \dfrac{2\pi}{\sqrt{37}/2} = \boxed{\dfrac{4\pi}{\sqrt{37}} \approx 2.066}$

b) $4x'' + 4x' + 37x = 0$

$x = e^{rt} \to (4r^2+4r+37)e^{rt} = 0$

$4r^2+4r+37 = 0$

$r = \dfrac{-4 \pm \sqrt{4^2 - 4\cdot 4\cdot 37}}{2\cdot 4} = \dfrac{-1 \pm \sqrt{-36}}{2} = \dfrac{-1 \pm 6i}{2}$

$= -\frac{1}{2} \pm 3i$, $e^{rt} = e^{(-\frac{1}{2}\pm 3i)t} = e^{-t/2}e^{\pm 3it}$

$= e^{-t/2}(\cos 3t \pm i \sin 3t)$

$\hookrightarrow$ real basis soln space: $e^{-t/2}\cos 3t, e^{-t/2}\sin 3t$

$$\boxed{x = e^{-t/2}(c_1\cos 3t + c_2 \sin 3t)}$$ gen soln

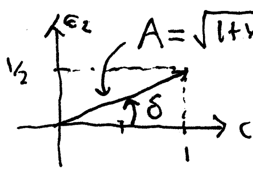
$$\boxed{k_1 = \tfrac{1}{2}, \tau_1 = 2 \quad \omega_1 = 3, T_1 = 2\pi/3 \approx 2.094}$$

c) $x' = -\frac{1}{2}e^{-t/2}(c_1\cos 3t + c_2\sin 3t) + e^{-t/2}(-3c_1\sin 3t + 3c_2\cos 3t)$

$x(0) = c_1 = 1$

$x'(0) = -\frac{1}{2}c_1 + 3c_2 = 1 \to c_2 = \frac{1}{3}(1+\frac{1}{2}(1)) = \frac{1}{2}$

$$\boxed{x = e^{-t/2}(\cos 3t + \tfrac{1}{2}\sin 3t)}$$



$A = \sqrt{1+1/4} = \sqrt{5}/2$

$\delta = \arctan 1/2$

$x = \underbrace{\frac{\sqrt{5}}{2}e^{-t/2}}\cos(3t - \arctan\frac{1}{2})$

$$\boxed{x = \pm\frac{\sqrt{5}}{2}e^{-t/2}}$$ envelope functions

$5\tau_1 = 5\cdot 2 = 10$ so $t = 0..10$

Note $5\tau_1/T_1 \approx 5$ one clearly sees 5 cycles!

d) $4x'' + 4x' + 37x = 36te^{-2t}$

for fun hand soln (not required):

$(D+2)^2(te^{-2t}) = 0$; $r = -2, m = 2$

$37\,[x_p = (c_3 + c_4 t)e^{-2t}]$

$4\,[x_p' = [c_4 - 2(c_3+c_4 t)]e^{-2t} = [c_4 - 2c_3 - 2c_4 t]e^{-2t}]$

$4\,[x_p'' = [-2c_4 - 2(c_4 - 2c_3 - 2c_4 t)]e^{-2t} = [-4c_4 + 4c_3 + 4c_4 t]e^{-2t}]$

$4x_p'' + 4x_p' + 37x_p = \begin{pmatrix} 37c_3 & + 37c_4 t \\ -8c_3 + 4c_4 & - 8c_4 t \\ +16c_3 - 16c_4 & + 16c_4 t \end{pmatrix} e^{-2t}$

$= (\underbrace{45c_3 - 12c_4}_{=0} + \underbrace{45c_4}_{=36} t)e^{-2t} = (0 + 36t)e^{-2t}$

$45c_3 - 12c_4 = 0 \longrightarrow c_3 = \frac{12}{45}c_4 = \frac{4}{15}(\frac{4}{5}) = \frac{16}{75}$

$45c_4 = 36 \to c_4 = \frac{36}{45} = \frac{4}{5}$

$x_p = (\frac{16}{75} + \frac{4}{5}t)e^{-2t} = \boxed{\frac{4}{75}(4+15t)e^{-2t}}$ maple agrees!

$x_h = \dots$ part b.

$x = x_h + x_p = e^{-\frac{t}{2}}(c_1\cos 3t + c_2 \sin 3t) + \frac{4}{75}(4+15t)e^{-2t}$

$x' = -\frac{1}{2}e^{-t/2}(c_1\cos 3t + c_2 \sin 3t) + e^{-t/2}(-3c_1\sin 3t + 3c_2\cos 3t) + \frac{4}{75}(\underbrace{-2(4+15t)+15}_{7-30t})e^{-2t}$

$x(0) = c_1 + 16/75 = 0 \longrightarrow c_1 = -16/75$

$x'(0) = -\frac{1}{2}c_1 + 3c_2 + 28/75 = 0 \to c_2 = \frac{1}{3}(\frac{1}{2}c_1 - \frac{28}{75}) = \frac{1}{3}(\frac{-8-28}{75})$

$= -12/75$

$$\boxed{x = \tfrac{4}{75}e^{-\frac{t}{2}}(-4\cos 3t - 3\sin 3t) + \tfrac{4}{75}(4+15t)e^{-2t}}$$ maple agrees!

$-16/75 \quad -4/25 \quad 16/75 \quad 4/5$

plot shows first peak (largest displacement) is around $t \approx 1$:

$0 = x' = -\frac{2}{75}e^{-t/2}(-4\cos 3t - 3\sin 3t) + \frac{4}{75}(-8-30t+15)e^{-2t} + \frac{4}{75}e^{-t/2}(12\sin 3t - 9\cos 3t)$

must solve numerically: only zero in $t = 0..2$ so

Maple: $\boxed{t = 1.0949, \quad x = 0.2573}$

e) $F = 45(2\cos 3t - 3\cos 3t)$

$37\,(x_p = c_3\cos 3t + c_4 \sin 3t)$

$4\,[x_p' = -3c_3 \sin 3t + 3c_4\cos 3t]$

$4\,[x_p'' = -9c_3\cos 3t - 9c_4 \sin 3t]$

$45(2\cos 3t - 3\sin 3t)$

$\|$

$4x_p'' + 4x_p' + 37x_p = \underbrace{[(37-36)c_3 + 12c_4]}_{=45(2)}\cos 3t + \underbrace{[-12c_3 + (37-36)c_4]}_{=45(-3)}\sin 3t$

$c_3 + 12c_4 = 45(2)$

$-12c_3 + c_4 = -45(3)$

$\begin{bmatrix} 1 & 12 \\ -12 & 1 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = 45\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \dfrac{45}{1+144}\begin{bmatrix} 1 & -12 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \dfrac{45}{145}\begin{bmatrix} 2+36 \\ 24-3 \end{bmatrix} = \dfrac{9}{29}\begin{bmatrix} 38 \\ 21 \end{bmatrix} = \dfrac{1}{29}\begin{bmatrix} 342 \\ 189 \end{bmatrix}$

$x_p = \frac{9}{29}(38\cos 3t + 21 \sin 3t)$

MAT 2705-01/02 14F Takehome Test 3 Answers (2)

1) e) continued

$$x = e^{-t/2}(c_1 \cos 3t + c_2 \sin 3t) + \frac{9}{29}(38 \cos 3t + 21 \sin 3t)$$

$$x' = e^{-t/2}(-3c_1 \sin 3t + 3c_2 \cos 3t) + \frac{9}{29}(-3\cdot 38 \sin 3t + 3\cdot 21 \cos 3t) - \frac{1}{2}e^{-t/2}(c_1 \cos 3t + c_2 \sin 3t)$$

$$x(0) = c_1 + \frac{9}{29}(38) = 0 \rightarrow c_1 = -\frac{9}{29}(38) = -342/29$$

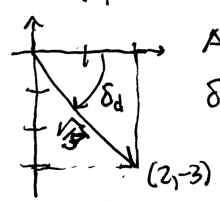
$$x'(0) = 3c_2 - \frac{1}{2}c_1 + \frac{9}{29}(3\cdot 21) \rightarrow c_2 = \frac{1}{3}\left(\frac{1}{2}c_1 - \frac{9}{29}(3\cdot 21)\right)$$

$$c_2 = \frac{1}{29}(-3\cdot 19 - 9\cdot 21) = -246/29$$

$$\boxed{x = \frac{1}{29}e^{-\frac{t}{2}}(-342 \cos 3t - 246 \sin 3t) + \frac{9}{29}(38 \cos 3t + 21 \sin 3t)}$$ Maple agrees!

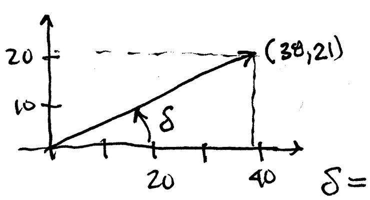
342 189

f) $F(t) = 45(2\cos 3t - 3\sin 3t)$: $T_d = \frac{2\pi}{3} \approx 2.09$



$A_d = 45\sqrt{13} \approx 162.25$

$\delta_d = -\arctan 3/2 \approx -56.3°$



$A = \frac{9}{29}\sqrt{38^2+21^2} = \frac{9}{29}\sqrt{1885} \approx 13.47$

$\delta = \arctan(21/38) \approx .505 \approx 28.9°$

$\delta - \delta_d = \arctan\left(\frac{21}{38}\right) + \arctan\left(\frac{3}{2}\right) \approx 85.2° \approx 0.237$ cycle

$$\frac{A}{A_d} = \frac{\frac{9}{29}\sqrt{1885}}{45\sqrt{13}} = \frac{1}{\sqrt{145}} = 0.0830$$

g) plot for $t = 0..5\tau_1 = 0..10$

The response (steady state) peaks should just be a hair less than a quarter cycle behind F. Plot them $t = -\pi/3..\pi/3$.

h)
$$\begin{aligned} 37\,[x_p &= c_3 \cos\omega t + c_4 \sin\omega t\,] \\ 4\,[x_p' &= -\omega c_3 \sin\omega t + \omega c_4 \cos\omega t\,] \\ 4\,[x_p'' &= -\omega^2 c_3 \cos\omega t - \omega^2 c_4 \sin\omega t\,] \end{aligned}$$

$$4x_p'' + 4x_p' + 37x_p = \underbrace{[(37-4\omega^2)c_3 + 4\omega c_4]}_{0} \cos\omega t + \underbrace{[-4\omega c_3 + (37-4\omega^2)c_4]}_{B_0\omega^2} \sin\omega t = B_0\omega^2 \sin\omega t$$

$$\begin{bmatrix} 37-4\omega^2 & 4\omega \\ -4\omega & 37-4\omega^2 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ B_0\omega^2 \end{bmatrix}$$

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{(37-4\omega^2)^2+16\omega^2}\begin{bmatrix} 37-4\omega^2 & -4\omega \\ 4\omega & 37-4\omega^2 \end{bmatrix}\begin{bmatrix} 0 \\ B_0\omega^2 \end{bmatrix} = \frac{B_0\omega^2}{(37-4\omega^2)^2+16\omega^2}\begin{bmatrix} -4\omega \\ 37-4\omega^2 \end{bmatrix}$$

(h) continued

$$\boxed{x_p = \frac{B_0\omega^2}{(37-4\omega^2)^2+16\omega^2}\left(-4\omega \cos\omega t + (37-4\omega^2)\sin\omega t\right)}$$

i)
$$\boxed{A(\omega)} = \frac{B_0\omega^2}{(37-4\omega^2)^2+16\omega^2}\sqrt{16\omega^2+(37-4\omega^2)^2}$$

$$= \boxed{B_0\omega^2\left((37-4\omega^2)^2+16\omega^2\right)^{-1/2}}$$

$$= B_0\omega^2\left(16\omega^4 - 280\omega^2 + 1369\right)^{-1/2}$$

$$\lim_{\omega\to\infty} A(\omega) = \lim_{\omega\to\infty} \frac{B_0\omega^2}{4\omega^2\left(1 - \underbrace{\frac{280}{16\omega^2}}_{\to 0} + \underbrace{\frac{1369}{16\omega^4}}_{\to 0}\right)^{1/2}} = \boxed{\frac{B_0}{4}}$$

$$a(3) = \frac{A(3)}{B_0\cdot 9} = \frac{1}{((37-4\cdot 9)^2 + 16\cdot 9)^{1/2}} = \frac{1}{\sqrt{145}}$$ ✓ yes agrees with f)

j)
$$0 = \frac{A'(\omega)}{B_0} = \frac{(16\omega^4 - 280\omega^2 + 1369)^{1/2}2\omega - \omega^2\frac{1}{2}(\dots)^{-1/2}(64\omega^3 - 560\omega)}{(\dots)^1}$$

num$\times(\,)^{1/2}$: $(16\omega^4 - 280\omega^2 + 1369)2\omega - \omega^2(32\omega^3 - 280\omega) = 0$

$2\omega(1369 - 280\omega^2 + 140\omega^2) = 0$

$2\omega(1369 - 140\omega^2) = 0$

$\omega \ge 0$: $\omega = 0,\ \sqrt{\frac{1369}{140}} = \frac{37}{70}\sqrt{35}$ $1369 = 37^2$

$$\boxed{\omega_p = \frac{37}{70}\sqrt{35} \approx 3.1270} = \frac{37}{2\sqrt{35}}$$

slightly bigger than $\omega_0 \approx 3.041$

$$\cancel{A(\omega_p)^{-2} = \left(37 - \frac{37^2}{4\cdot 35}\cdot 4\right)^2 + 16\,\frac{37^2}{4(35)}}$$

$$\cancel{= \left(37 - \frac{37^2}{35}\right)^2 + \frac{4\cdot 37^2}{35} = \frac{37}{35}\left(\left(-\frac{37}{35}\right)^2 + \frac{4\cdot 37}{35}\right)}$$

$$\cancel{= \frac{37}{35}\left(\left(\frac{4}{35}\right)^2 + \frac{4\cdot 37}{35}\right) = \frac{4\cdot 37}{35^3}(4 + 37\cdot 35)}$$ almost but not quite, punt!

$$A(\omega_p) \overset{\text{maple}}{=\!=} \frac{37}{48}B_0$$

slightly bigger than:

$$\frac{A(\omega_p)}{B_0/4} = \frac{37}{48}\cdot 4 = \frac{37}{12} \approx \boxed{3.083} \leftrightarrow Q \approx 3.041$$

k)
$$\frac{A(\omega)}{B_0/4} = \frac{4\omega^2}{\sqrt{(37-4\omega^2)^2+16\omega^2}}$$

$$\frac{A(\omega_0)}{B_0/4} = \frac{\sqrt{37}}{2} = Q\,!$$

MAT2705-01/02 14F Takehome test 3 Answers (3)

② a) $(k_1,k_2,k_3) = \left(\frac{20}{100}, \frac{20}{40}, \frac{20}{40}\right) = \left(\frac{1}{5}, \frac{1}{2}, \frac{1}{2}\right)$

$$\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}' = \underbrace{\begin{bmatrix} -1/5 & 0 & 1/2\\ 1/5 & -1/2 & 0\\ 0 & 1/2 & -1/2\end{bmatrix}}_{A}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}, \quad \begin{bmatrix} x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix} = \begin{bmatrix} 9\\ 0\\ 9\end{bmatrix}$$

b) $$|A-\lambda I| = \begin{vmatrix} -1/5-\lambda & 0 & +1/2\\ 1/5 & -1/2-\lambda & 0\\ 0 & 1/2 & -1/2-\lambda\end{vmatrix} \overset{\text{maple}}{=} -\lambda^3 - \frac{6}{5}\lambda^2 - \frac{9}{20}\lambda$$

$$= -\frac{\lambda}{20}(20\lambda^2+24\lambda+9) = 0 \to \lambda = 0, -\frac{3}{5} \pm \frac{3}{10}i$$

$$\lambda = 0: \begin{bmatrix} -1/5 & 0 & 1/2\\ 1/5 & -1/2 & 0\\ 0 & 1/2 & -1/2\end{bmatrix} \xrightarrow{\text{Maple}} \begin{array}{c} \begin{matrix} L & L & F \end{matrix}\\ \begin{bmatrix} 1 & 0 & -5/2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}\end{array}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

$x_3 = t \quad x_1 = 5/2\,t$
$\qquad\quad x_2 = t$ $$\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix} 5/2\,t\\ t\\ t\end{bmatrix} = t\underbrace{\begin{bmatrix} 5/2\\ 1\\ 1\end{bmatrix}}_{\vec b_1}$$

$$\lambda = -\frac{3}{5}+\frac{3}{10}i: \begin{bmatrix} -\frac{1}{5}+\frac{3}{5}-\frac{3}{10}i & 0 & 1/2\\ 1/5 & -\frac{1}{2}+\frac{3}{5}-\frac{3}{10}i & 0\\ 0 & 1/2 & -\frac{1}{2}+\frac{3}{5}-\frac{3}{10}i\end{bmatrix}$$

$$= \begin{bmatrix} \frac{4-3i}{10} & 0 & 1/2\\ 1/5 & \frac{-2-3i}{10} & 0\\ 0 & 1/2 & \frac{-2-3i}{10}\end{bmatrix} \xrightarrow{\text{Maple}} \begin{array}{c} \begin{matrix} L & L & F \end{matrix}\\ \begin{bmatrix} 1 & 0 & (4+3i)/5\\ 0 & 1 & (1-3i)/5\\ 0 & 0 & 0\end{bmatrix}\end{array}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$

$x_3 = t \quad x_1 = -(4+3i)/5\,t$
$\qquad\quad x_2 = (-1+3i)/5$ $$\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} = t\underbrace{\begin{bmatrix} -(4+3i)/5\\ (-1+3i)/5\\ 1\end{bmatrix}}_{\vec b_2} \quad \vec b_2 \xrightarrow{\text{c.c.}} \vec b_3 = \overline{\vec b_2}$$

$$B = \begin{bmatrix} 5/2 & -(4+3i)/5 & -(4-3i)/5\\ 1 & (-1+3i)/5 & (-1-3i)/5\\ 1 & 1 & 1\end{bmatrix}$$

$$\vec x = B\vec y,\ \vec y = B^{-1}\vec x \to \vec x' = Ax \to \vec y' = A_B\vec y$$

$$A_B = B^{-1}AB = \operatorname{diag}\left(0, -\frac{3}{5}+\frac{3}{10}i, -\frac{3}{5}-\frac{3}{10}i\right)$$

$$y_i' = \lambda_i y_i \to y_i = c_i e^{\lambda_i t} \to \vec x = \sum_{i=1}^{3} c_i e^{\lambda_i t}\vec b_i$$

$$e^{\left(-\frac{3}{5}+\frac{3}{10}i\right)t}\begin{bmatrix} -(4+3i)/5\\ (-1+3i)/5\\ 1\end{bmatrix} = e^{-\frac{3}{5}t}\begin{bmatrix} -\frac{1}{5}(c+is)(4+3i)\\ \frac{1}{5}(c+is)(-1+3i)\\ \cos\frac{3t}{10} + i\sin\frac{3t}{10}\end{bmatrix}$$

$$= \underbrace{e^{-\frac{3t}{5}}\begin{bmatrix} -\frac{4}{5}\cos\frac{3t}{10} + \frac{3}{5}\sin\frac{3t}{10}\\ -\frac{1}{5}\cos\frac{3t}{10} - \frac{3}{5}\sin\frac{3t}{10}\\ \cos\frac{3t}{10}\end{bmatrix}}_{\vec X_1} + i\,\underbrace{e^{-\frac{3t}{5}}\begin{bmatrix} -\frac{4}{5}\sin\frac{3t}{10} - \frac{3}{5}\cos\frac{3t}{10}\\ -\frac{1}{5}\sin\frac{3t}{10} + \frac{3}{5}\cos\frac{3t}{10}\\ \sin\frac{3t}{10}\end{bmatrix}}_{\vec X_2}$$

b) continued

$$\boxed{\vec x = c_1\begin{bmatrix} 5/2\\ 1\\ 1\end{bmatrix} + a\vec X_1 + b\vec X_2}$$ general soln

c) $$\vec x(0) = c_1\begin{bmatrix} 5/2\\ 1\\ 1\end{bmatrix} + a\begin{bmatrix} -4/5\\ -1/5\\ 1\end{bmatrix} + b\begin{bmatrix} -3/5\\ 3/5\\ 0\end{bmatrix}$$

$$= \begin{bmatrix} 5/2 & -4/5 & -3/5\\ 1 & -1/5 & 3/5\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix} c_1\\ a\\ b\end{bmatrix} = \begin{bmatrix} 9\\ 0\\ 9\end{bmatrix}$$

$$\begin{bmatrix} c_1\\ a\\ b\end{bmatrix} = \begin{bmatrix} \text{"} \end{bmatrix}^{-1}\begin{bmatrix} 9\\ 0\\ 9\end{bmatrix} \overset{\text{Maple}}{=} \begin{bmatrix} 4\\ 5\\ -5\end{bmatrix}$$

$$\vec x = 4\begin{bmatrix} 5/2\\ 1\\ 1\end{bmatrix} + e^{-\frac{3t}{5}}\begin{bmatrix} -4c+3s\\ -c-3s\\ 5c\end{bmatrix} - e^{-\frac{3t}{5}}\begin{bmatrix} -4s-3c\\ -s+3c\\ 5s\end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix} 10 - e^{-3t}\cos\frac{3}{10}t + 7e^{-\frac{3t}{5}}\sin\frac{3t}{10}\\ 4 - 4e^{-\frac{3t}{5}}\cos\frac{3t}{10} - 2e^{-\frac{3t}{5}}\sin\frac{3t}{10}\\ 4 + 5e^{-\frac{3t}{5}}\cos\frac{3t}{10} - 5e^{-\frac{3t}{5}}\sin\frac{3t}{10}\end{bmatrix}}$$

d) $$\lim_{t\to\infty}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix} 10\\ 4\\ 4\end{bmatrix}$$

e) $\tau = (3/5)^{-1} = \frac{5}{3} \approx 1.67$

$5\tau = \frac{25}{3} \approx 8.33$

plot $t = 0..10$?

MAT2705-01/02 14F Takehome Test 3 Answers (4)

③ a) $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -10 & 1 \\ 2 & -11 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$A \hookrightarrow \vec{x}\,' = A\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$0 = |A - \lambda I| = \begin{vmatrix} -10-\lambda & 1 \\ 2 & -11-\lambda \end{vmatrix} = (10+\lambda)(11+\lambda) - 2 = \lambda^2 + 21\lambda + 108 = 0$

$\lambda = \frac{-21 \pm \sqrt{21^2 - 4\cdot 4\cdot 27}}{2} = \frac{-21 \pm \sqrt{9\cdot 7^2 - 9\cdot 3\cdot 16}}{2} = \frac{-21 \pm 3\sqrt{49-48}}{2}$!

$= \frac{-21 \pm 3}{2} = -12, -9 \longrightarrow -9, -12$ increasing absolute value.

$\lambda = -9$: $A + 9I = \begin{bmatrix} 9-10 & 1 \\ 2 & 9-11 \end{bmatrix} \leftrightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} L & F \end{array} \\ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t, \; x_1 = x_2 = t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \vec{b}_1$

$\lambda = -12$: $A + 12I = \begin{bmatrix} 12-10 & 1 \\ 2 & 12-11 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \to \begin{array}{c} \begin{array}{cc} L & F \end{array} \\ \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t, \; x_1 = -\tfrac{1}{2}x_2 = -\tfrac{1}{2}t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -\tfrac{1}{2} \\ 1 \end{bmatrix}$

$\vec{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ double for lowest integer eigenvector

$\boxed{\lambda = -9, -12 \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}} \to A_B = B^{-1}AB = \begin{bmatrix} -9 & 0 \\ 0 & -12 \end{bmatrix}$

b) $\vec{x} = B\vec{y}, \; \vec{y} = B^{-1}\vec{x}$:

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \longrightarrow$

c) $\vec{x}\,' = A\vec{x} \to B^{-1}[(B\vec{y})' = A(B\vec{y})]$

$\to \vec{y}\,' = B^{-1}AB\,\vec{y} = A_B\vec{y}$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -12 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -9y_1 \\ -12y_2 \end{bmatrix}$

$y_1' = -9y_1, \; y_1 = c_1 e^{-9t} \qquad y_1(0) = c_1 = 4$

$y_2' = -12y_2, \; y_2 = c_2 e^{-12t} \qquad y_2(0) = c_2 = -2$

$y_1 = 4e^{-9t}, \; y_2 = -2e^{-12t}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4e^{-9t} \\ -2e^{-12t} \end{bmatrix} = 4e^{-9t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2e^{-12t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 4e^{-9t} + 2e^{-12t} \\ 4e^{-9t} - 4e^{-12t} \end{bmatrix}$ IVP soln

faster decay mode: $\tau_1 = 1/9$, $5\tau_1 = 5/9$ — slower decay mode: $\tau_2 = 1/12$

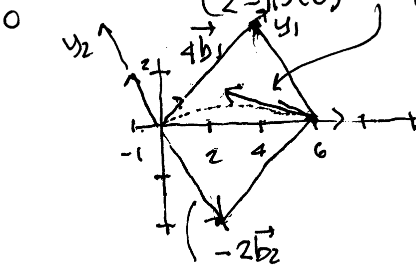
$\hookrightarrow t = 0..1/2$ plot range

d) $\langle 0,6 \rangle = 4\vec{b}_1 - 2\vec{b}_2 = \langle 4,4 \rangle + \langle 2,-4 \rangle$

projection parallelogram vertices:

[[0,0], [4,4], [6,0], [2,−4]]

$\vec{x}\,'(0) = A\vec{x}(0) = \begin{bmatrix} -10 & 1 \\ 2 & -11 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = 12\begin{bmatrix} -5 \\ 1 \end{bmatrix}$



collapses faster in y_1 direction so concave down as heads for origin!

e) $x_2 = 4e^{-9t} - 4e^{-12t}$

$0 = x_2' = 4(-9e^{-9t} - (-12)e^{-12t})$

$= -12(3e^{-9t} - 4e^{-12t})$

$[3e^{-9t} = 4e^{-12t}]\, e^{12t}/3$

$e^{3t} = 4/3 \qquad t = \boxed{\tfrac{1}{3}\ln(4/3) \approx 0.0959}$

$x_2 = 4(e^{-9(\frac{1}{3}\ln(4/3))} - e^{-12(\frac{1}{3}\ln(4/3))})$

$= 4\left(\left(\tfrac{4}{3}\right)^{-3} - \left(\tfrac{4}{3}\right)^{-4}\right)$

$= 4\left(\left(\tfrac{3}{4}\right)^3 - \left(\tfrac{3}{4}\right)^4\right) = 4\left(\tfrac{3}{4}\right)^3\left(1 - \tfrac{3}{4}\right)$

$= \left(\tfrac{3}{4}\right)^3 \approx \boxed{\tfrac{27}{64} \approx 0.4219}$