MAT2705-01/02 14F Take Home Test 3 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions, matrix inverses, plotting and root finding without showing intermediate steps.** ***Print*** the requested 9 technology plots, labeling them and **annotating them appropriately *by hand*** and attach to the end of your test. All differential equations should be solved "by hand" unless otherwise specified.

1. The displacement $x(t)$ of an underdamped harmonic oscillator system satisfies

$$m\,x''(t) + c\,x'(t) + K\,x(t) = F(t).$$

Let $m=4$, $c=4$, $K=37$ and the initial conditions $x(0)=x_0$, $x'(0)=v_0$.

a) Express this DE in standard form with a unit coefficient of the second derivative term. What are the natural frequency ω_0, natural decay time τ_0, the quality factor $Q=\omega_0\tau_0$ and the period $T_0=2\pi/\omega_0$ for this system? (Give both exact and numeric values to 3 decimal places.)

Consider the following driving force functions $F(t)$:

b) $F(t)=0$. Find the general solution of the differential equation. What is the frequency ω_1, decay time τ_1 and the period $T_1=2\pi/\omega_1$ for this decaying sinusoidal solution? (Give both exact and numeric values to 3 decimal places.

c) $F(t)=0$. Solve the above initial value problem for $x_0=1$, $v_0=1$. Evaluate the two envelope functions for the exponentially decaying sinusoidal function and **plot (#1)** all three together for $t=0..5\,\tau_1$, fully labeling the axes and three curves. [You should see a number of oscillations before convergence to the horizontal axis. How many do you see in the chosen window? Does this agree with your value of T_1?]

d) $F(t)=36\,t\,e^{-2t}$. For the initial system at rest at equilibrium, find the maximum displacement from equilibrium and the time at which it occurs numerically to 4 decimal places. Find the solution using Maple and find the first extremal point on the curve using technology. **Plot (#2)** your solution for $t=0..5\,\tau_p$, where τ_p is the characteristic time for the exponential factor here. Compare your coordinate values to your graph, annotating the maximal point on the graph.

e) $F(t)=45\,(2\cos(3t)-3\sin(3t))$. Find the initial value problem solution for $x_0=0=v_0$ by hand using matrix methods (but check with Maple!).

f) Evaluate the values of the amplitude A and phase shift δ for the steady state solution (the part of the solution which remains after the transient has died away). Evaluate the relative phase shift δ_{rel} by subtracting the phase shift δ_d of the driving function. Express this relative phase shift in radians, degrees and cycles (divide radians by 2π). What is the ratio $a=A/A_d$ of the response (steady state) amplitude to the driving force amplitude A_d of $F(t)$? (amplification factor)

.

g) Make a single **plot (#3)** in an appropriate viewing window ($t\geq 0$!) showing both the solution function and the steady state solution until they merge. In a separate plot for comparison with the driving sine function, **plot (#4)** both the steady state solution and the driving function rescaled to have the same amplitude as the steady state solution in order to see how the peaks of the steady state solution compare to the peaks of the driving function, using the window $t=-T_d/2..T_d/2$, where T_d is the period of the driving function. Does your plot agree with your calculated relative phase shift angle δ_{rel} (does the steady state solution lead ahead in time [peaks earlier] or lag behind in time [peaks later] the driving function by a corresponding amount)? Explain.

h) $F(t)=B_0\omega^2\sin(\omega t)$, $B_0>0$, $\omega>0$, $x_0=0=v_0$.

Explore resonance for this system by finding the steady state solution by hand, where the frequency ω of the driving force function is a parameter.

i) Evaluate the steady state amplitude function $A(\omega)$ and its limit $A(\infty)=\lim_{\omega\to\infty}A(\omega)$. Does your value $a(3)$ of the amplification factor $a(\omega)=A(\omega)/\left(B_0\omega^2\right)$ agree with your value for part f) as it should?

j) Use calculus to find the exact and numerical values of both the frequency ω_p and the amplitude ratio $A(\omega_p)/A(\infty)$ where it has its peak value for $\omega\geq 0$. What is the numerical value of $A(\omega_p)/A(\infty)$ and how does the its value compare to the quality factor Q?

k) **Plot (#5)** the amplitude function $A(\omega)/A(\infty)$ in an appropriate window (showing the limiting behavior of the entire function for $\omega \geq 0$ and hand annotate on your axes the values of ω_p, ω_0 and corresponding amplitudes and indicate the points on the curve which correspond to ω_0 and ω_p.

2. $x_1'(t) = -k_1\, x_1(t) + k_3 x_3(t), x_2'(t) = k_1 x_1(t) - k_2 x_2(t), x_3'(t) = k_2 x_2(t) - k_3 x_3(t),$
$x_1(0) = 9, x_2(0) = 0, x_3(0) = 9$; $k_i = r/V_i, r = 20$, $(V_1, V_2, V_3) = (100, 40, 40)$.

a) Write this closed 3 tank mixing tank system of differential equations for the vector variable $\vec{x} = \langle x_1, x_2, x_3 \rangle$ AND its initial conditions in explicit matrix form (showing all components, not just vector symbols) and identifying the coefficient matrix A.
b) Use the eigenvector approach to find its general solution by hand, showing all steps.
c) Find the IVP solution, using matrix methods showing all steps. Make sure it agrees with Maple's solution.
d) Evaluate the asymptotic solution $\vec{x}_\infty$ for $t \to \infty$, which is the equilibrium solution approached by your solution.
e) **Plot (#6)** the three solution curves together with their horizontal asymptotes in an appropriate viewing window ($t \geq 0$!) that shows them reaching and settling down to those equilibrium values without compressing the interesting part before reaching that limit. Label each curve. Comment about your choice of viewing window in relation to the characteristic decay time of the problem.

3. $x_1'(t) = -10\, x_1(t) + x_2(t), x_2'(t) = 2\, x_1(t) - 11\, x_2(t), x_1(0) = 6, x_2(0) = 0.$

a) Identify the coefficient matrix and by hand find a new basis for R^2 consisting of eigenvectors $\vec{b}_1, \vec{b}_2$ of this matrix using the standard hand recipe, rescaling them to be the **minimal integer** eigenvectors. Order the real eigenvalues $\lambda_1 \geq \lambda_2$ by decreasing value (increasing absolute value). Identify $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$.
b) Evaluate the new coordinates $\langle y_1, y_2 \rangle$ of the point $\langle x_1, x_2 \rangle = \langle 6, 0 \rangle$ with respect to this basis of eigenvectors.
c) Solve this initial value problem by hand, showing all steps.
d) Use technology to **plot (#7)** a directionfield for this DE with the solution curve through the single initial data point, and (by hand if necessary) include the lines through the two eigenvectors representing the two subspaces of eigenvectors. Choose an appropriate window that shows everything clearly without wasting additional window space. By hand label these lines by their new coordinate labels, draw in and label the eigenvectors and the initial data vector $\vec{x}(0)$ themselves as arrows, and include the parallelogram projection of the latter vector onto the new coordinate axes, i.e., the parallelogram parallel to the new coordinate axes with the initial data vector as the main diagonal. Do the projections along the coordinate axes agree with the values you found for the new coordinates of this vector? Explain. Does your directionfield correspond to the eigenvectors you have drawn? Explain why.
e) **Plot (#8)** the two variables versus t for an appropriate viewing window for this initial value problem. Explain your window choice.The function x_2 has a global maximum (for $t > 0$) , obvious in a separate **plot (#9)** of this variable. Find its coordinates and the corresponding value of t exactly and approximately (Maple is helpful to evaluate $(x_2)(t_{max})$ exactly). Annotate your diagram to show this point.

Advice. When in doubt about how much work to show, show more. Explain using words if it helps. Think of this take-home test as an exercise in "writing intensive" technical expression. Try to impress bob as though it were material for a job interview. In a real world technical job, you need to be able to write coherent technical reports that other people can follow.

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have read the long instructions on the class web page. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: