MAT2705-01/02 F14 Test 1 Answers

① a) $\frac{dT}{dt} = -k(T-325)$

$T(0) = 50, \quad T(1.5) = 125, \quad T(t) = 165?$

initial condition, secondary condition, prediction

$\int \frac{dT}{T-325} = \int -k\,dt$ separate and integrate

$\ln|T-325| = -kt + C_1$

$\underbrace{e^{\ln|T-325|}}_{|T-325|} = e^{-kt+C_1} = e^{C_1}e^{-kt}$

$T-325 = \underbrace{\pm e^{C_1}}_{C} e^{-kt} = Ce^{-kt}$

$\boxed{T = 325 + Ce^{-kt}}$ general soln, Maple agrees!

$50 = T(0) = 325 + C \rightarrow C = 50 - 325 = -275$

$\boxed{T = 325 - 275e^{-kt}}$ IVP soln, Maple agrees!

$125 = T(3/2) = 325 - 275e^{-\frac{3}{2}k}$

$275e^{-3/2k} = 325 - 125 = 200$

$e^{-\frac{3}{2}k} = \frac{200}{275} = \frac{8}{11}, \quad e^{\frac{3}{2}k} = \frac{11}{8}$

c) $\boxed{k = \frac{2}{3}\ln\frac{11}{8} \approx 0.212302}$

$\boxed{\tau = \frac{1}{k} = \frac{3}{2}\frac{1}{\ln 11/8} \approx 4.71}$

$165 = T(t) = 325 - 275e^{-kt}$

$275e^{-kt} = 325 - 165 = 160$

$e^{-kt} = \frac{160}{275} = \frac{32}{55}$

$e^{kt} = \frac{55}{32}, \quad t = \frac{1}{k}\ln 55/32$

$= \frac{3}{2}\frac{\ln 55/32}{\ln 11/8}$

≈ 2.55 hr

≈ 2hr $+ .55(60)$ min ≈ 2hr 33 min

$\boxed{\text{bob has to wait 2 hours and 33 minutes for the turkey thigh to reach } 165^\circ}$

c) $T(\tau) \approx T(4.71) \approx 223.8^\circ$

② a) $2\frac{dx}{dt} = 10 - \frac{4}{9+4t}x, \; x(0) = 0 \xrightarrow{\text{Maple}}$

$\boxed{x(t) = \frac{15}{2} + \frac{10}{3}t + \frac{15}{2(9+4t)^{1/2}}}$

② b) $\frac{dx}{dt} = 5 - \frac{2}{9+4t}x \rightarrow \boxed{\frac{dx}{dt} + \frac{2}{9+4t}x = 5}$ standard linear form

c) $\int \frac{2}{9+4t}dt = 2\cdot\frac{1}{4}\ln|9+4t| = \frac{1}{2}\ln(9+4t), \; t \geq 0$

$e^{\frac{1}{2}\ln(9+4t)} = (9+4t)^{1/2}$ integrating factor

d) $(9+4t)^{1/2}\left(\frac{dx}{dt} + \frac{2}{9+4t}x\right) = 5(9+4t)^{1/2}$

$\frac{d}{dt}\left(x(9+4t)^{1/2}\right) = 5(9+4t)^{1/2}$

$x(9+4t)^{1/2} = \int 5(\underbrace{9+4t}_{u})^{1/2}\underbrace{dt}_{\frac{du}{4}} = \frac{5}{4}\int u^{1/2}du = \frac{5}{4}\frac{u^{3/2}}{3/2} + C$

$du = 4dt$

$= \frac{5}{6}(9+4t)^{3/2} + C$

$x = \left[\frac{5}{6}(9+4t)^{3/2} + C\right]/(9+4t)^{1/2}$

$= \frac{5}{6}(9+4t) + \frac{C}{(9+4t)^{1/2}}$

$\boxed{x = \frac{15}{2} + \frac{10}{3}t + \frac{C}{(9+4t)^{1/2}}}$ gen soln

e) $10 = x(0) = \frac{15}{2} + \frac{C}{9^{1/2}} = \frac{15}{2} + \frac{C}{3} \rightarrow C = 3\left(10 - \frac{15}{2}\right) = \frac{15}{2}$

IVP soln: $\boxed{x = \frac{15}{2} + \frac{10}{3}t + \frac{15}{2(9+4t)^{1/2}}} = \frac{5}{6}(9+4t) + \frac{15}{2(9+4t)^{1/2}}$

f) $x(4) = \frac{15}{2} + \frac{40}{3} + \frac{15}{2(25)^{1/2}} = \frac{15}{2} + \frac{40}{3} + \frac{3}{2} = 9 + \frac{40}{3} = \boxed{\frac{67}{3}}$

$= 22\tfrac{1}{3} \approx \boxed{22.3}$

g) $x = \frac{5}{6}(9+4t) + \frac{15}{2(9+4t)^{1/2}}, \quad \frac{dx}{dt} = \frac{10}{3} + \frac{15}{2}\left(-\frac{1}{2}\right)(9+4t)^{-3/2}(4)$

$= \frac{10}{3} - 15/(9+4t)^{3/2}$

$2\left(\frac{10}{3} - \frac{15}{(9+4t)^{3/2}}\right) = 10 - \frac{4}{9+4t}\left(\frac{5}{6}(9+4t) + \frac{15}{2(9+4t)^{1/2}}\right)$

$\frac{20}{3} - \frac{30}{(9+4t)^{3/2}} = 10 - \frac{10}{3} - \frac{30}{(9+4t)^{3/2}}$

$= \frac{20}{3} - \frac{30}{(9+4t)^{3/2}}$ ✓

alternative expression: $-\frac{4}{9+4t}\left(\frac{15}{2} + \frac{10}{3}t\right) = -\frac{4}{9+4t}\left(\frac{45+20t}{6}\right)$

$= -\frac{2}{(9+4t)3}5(9+4t) = -\frac{10}{3}$

h) $C(t) = \left[\frac{5}{6}(9+4t) + \frac{15}{2(9+4t)^{1/2}}\right]/(9+4t) = \frac{5}{6} + \frac{15}{2(9+4t)^{3/2}} = 1$

$\frac{15}{2(9+4t)^{3/2}} = \frac{1}{6} \rightarrow \frac{90}{2} = (9+4t)^{3/2} \rightarrow \left(\frac{90}{2}\right)^{2/3} = 9+4t$

$t = \frac{1}{4}\left(\frac{90}{2}\right)^{3/2} - \frac{9}{4} \approx 0.9128 \rightarrow 0.913$ but only makes sense to simply solve numerically with technology

$C(0) \approx 1.11, \; C(4) \approx 0.89 \rightarrow C(t) = 1$ in between

MAT2705-01/02 14F Test 1 Print Name (Last, First) ______________________________|__

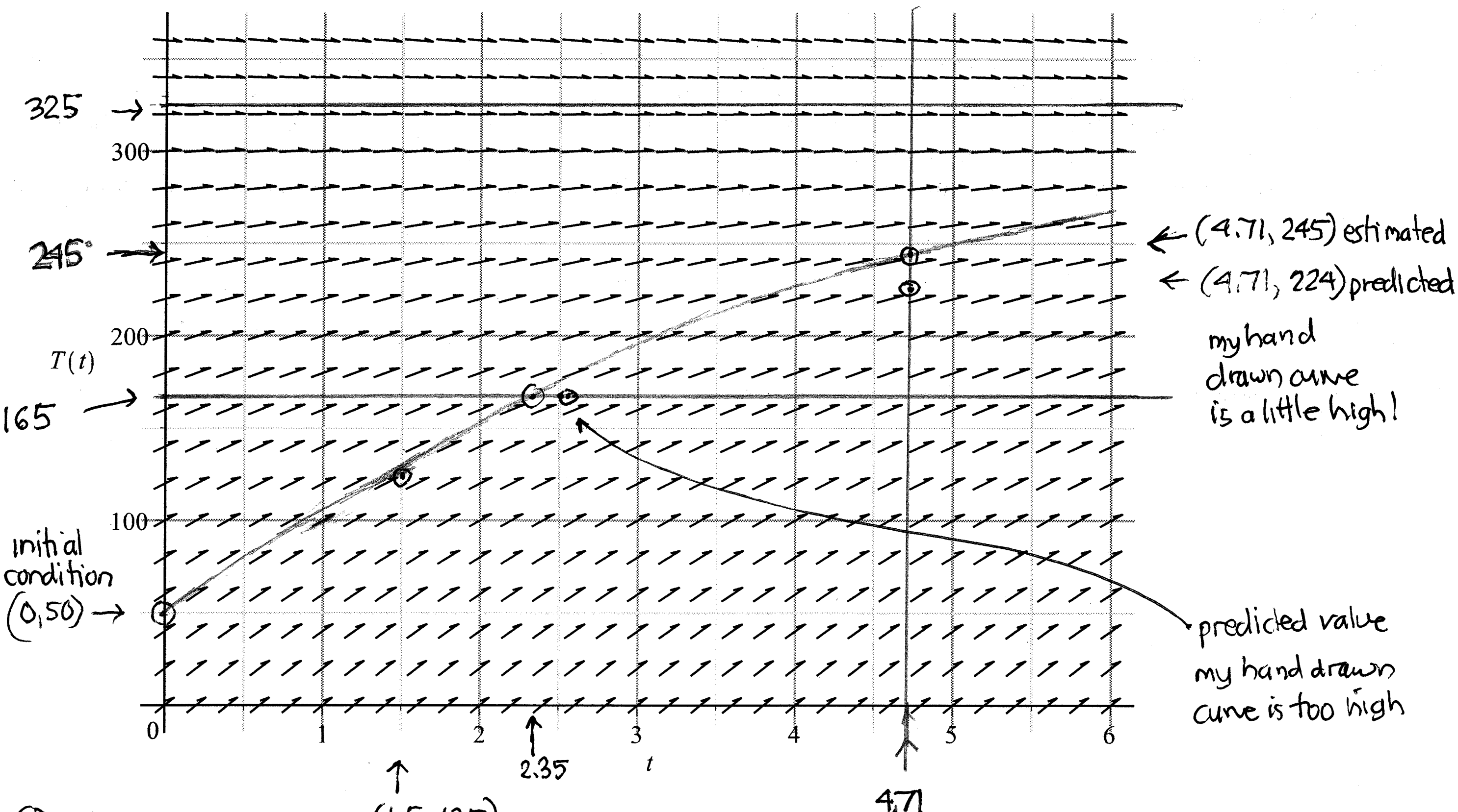
Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). [Recall you need $y'(t), y(t)$ instead of y', y in your differential equation for an unknown variable y for Maple to interpret the prime as a t derivative.]

1. $\dfrac{dT}{dt} = -k\,(T-325)$

a) Bob pulls a turkey from the refrigerator at 50 degrees and jams it into a 325 degree oven. After 1.5 hours the instant read thermometer jabbed into the turkey thigh reaches 125 degrees. If this turkey thigh obeys Newton's law of cooling/heating, how long does bob have to wait till the minimum recommended temperature of 165 degrees is reached? [Use the separable solution technique to solve this DE. Show every step of the process clearly. Answer this word problem with a complete English sentence which can be directly compared to a clock. Boxit. During the process keep things exact using 3/2 for 1.5 and don't introduce any decimal points long enough for you reach exact values for k and τ requested below.]

b) Using the slope field below, locate the initial data point and the secondary data point by circled dots. Then make a rough hand sketch of your solution, labeling on your sketch the initial and secondary data points given above, and including the horizontal lines corresponding to 325, 165 and 125 degrees (use a piece of paper as a straight edge?). Is your hand drawn curve consistent with your answer to the question posed in part a)? Explain.

c) What is the exact value of k and its numerical value to 6 decimal places? What is the exact value of the corresponding characteristic time τ (with units) and its numerical value to 3 significant digits? Mark this on the time axis and read off the temperature from your hand plot and compare with your predicted value at $t=\tau$. Comment.



① b) c) graph work:

my hand drawn curve is a bit high here too

But these are relatively small differences!

(turn the page)