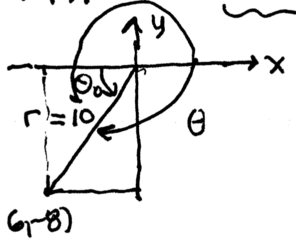
MAT2500-01 13S Final Exam Answers (1)

① $(x,y,z) = (-6,-8,-10)$



two choices for θ from reference angle θ_0

$\langle -6,-8\rangle = -2\langle 3,4\rangle$

3-4-5 triangle. $|\langle 3,4\rangle| = 5$

$\boxed{r = 2\cdot 5 = 10}$ $\left(= \sqrt{6^2+8^2} = \sqrt{36+64} = \sqrt{100} = 10\right)$

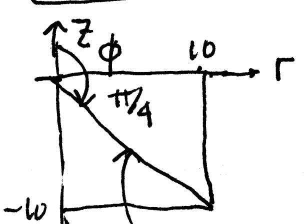
$\tan\theta_0 = \frac{4}{3}$, $\theta_0 = \arctan\frac{4}{3}$

$\theta = -\pi + \arctan\frac{4}{3}$ (negative obtuse) $\approx -126.9°$

or

$\theta = \pi + \arctan 4/3 \approx 233.1°$

$\boxed{z = -10}$



$\phi = \frac{\pi}{2} + \arctan(1)$

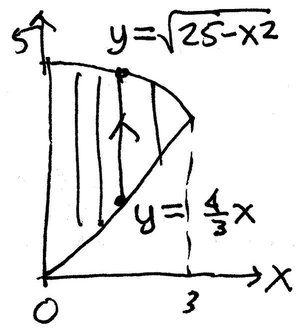
$= \frac{\pi}{2} + \frac{\pi}{4} = \boxed{\frac{3\pi}{4} = 135°}$

$\boxed{\rho = 10\sqrt{2}}$ $\left(= \sqrt{10^2+10^2} = \sqrt{10^2\cdot 2} = 10\sqrt{2}\right)$

a) cylindrical: (r,θ,z)

b) spherical: (ρ,ϕ,θ)

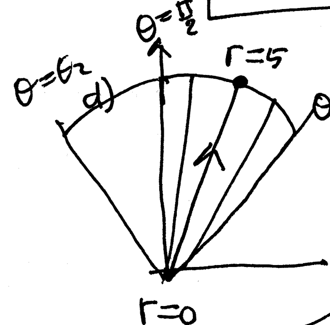
② c)



$A = 2\int_0^3 \int_{\frac{4}{3}x}^{\sqrt{25-x^2}} 1\,dy\,dx$

$= \boxed{\frac{25}{2}\pi - 25\arctan\left(\frac{4}{3}\right) \approx 16.0875}$

d)



$A = \int_{\theta_1}^{\theta_2}\int_0^5 1\, r\,dr\,d\theta$ = same as above...

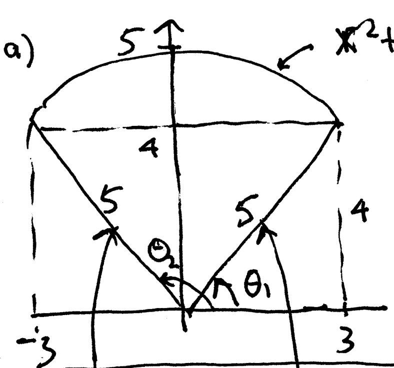
arctan $\frac{4}{3}$

by hand $= \int_{\theta_1}^{\theta_2} 1\,d\theta \int_0^5 r\,dr$ $\left.\frac{r^2}{2}\right|_0^5$

$= (\theta_2 - \theta_1)\cdot\frac{25}{2}$

$= \frac{25}{2}\pi - 25\arctan\frac{4}{3}$ ✓

② a)



$\boxed{y = -\frac{4}{3}x \quad x=-3..0}$ $\boxed{y = \frac{4}{3}x \quad x = 0..3}$

C_3: $\boxed{\theta = \theta_2,\ r = 0..5}$

C_1: $\boxed{\theta = \theta_1,\ r = 0..5}$

C_2: $\boxed{x = -3..3}$ C_2: $\boxed{\theta = \theta_1..\theta_2}$

$x^2+y^2 = 25 \rightarrow \boxed{y = \sqrt{25-x^2}} \Leftrightarrow \boxed{r=5}$

$\tan\theta_1 = 4/3$ $\boxed{\theta_1 = \arctan\frac{4}{3}}$ $\approx 53.1°$

$\boxed{\theta_2 = \pi - \arctan 4/3} \approx 126.9°$

b) $\Delta\theta = \theta_2 - \theta_1 = \boxed{\pi - 2\arctan 4/3 \approx 73.7°}$

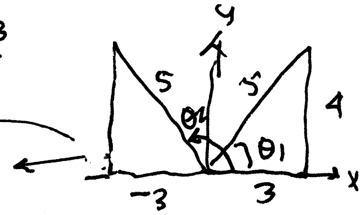
$\boxed{A = \frac{1}{2}25(\pi - 2\arctan 4/3) \approx 16.0875}$

MAT2500-01 13S Final Exam Answers (2)

2 e)

$$A_y = \iint y\, dA = \int_{\theta_1}^{\theta_2}\int_0^5 (r\sin\theta)\, r\, dr\, d\theta = \underbrace{\int_{\theta_1}^{\theta_2}\sin\theta\, d\theta}_{-\cos\theta\big|_{\theta_1}^{\theta_2}} \underbrace{\int_0^5 r^2 dr}_{r^3/3\big|_0^5}$$

$$= \underbrace{(-\cos\theta_2 + \cos\theta_1)}_{\underbrace{+\cos\theta_1}_{2\cos\theta_1 = 2\left(\frac{3}{5}\right)}} \frac{5^3}{3}$$

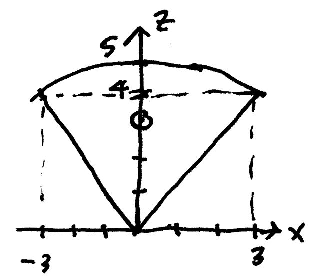


$\cos\theta_1 = \frac{3}{5}$

$\cos\theta_2 = -\frac{3}{5}$

$$= 2\left(\frac{3}{5}\right)\frac{5^3}{3} = 50$$

$$\bar{y} = \frac{A_y}{A} = \boxed{\frac{50}{\frac{25}{2}\pi - 25\arctan 4/3} \approx 3.108}$$



f) $\vec{F}(\vec{r}) = \frac{1}{2}\langle -y, x\rangle$

C_1: $\vec{r}(t) = \langle t, \frac{4}{3}t\rangle$ $t = 0..3$

$\vec{r}'(t) = \langle 1, 4/3\rangle$

$\vec{F}(\vec{r}(t)) = \frac{1}{2}\langle -\frac{4}{3}t, t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = \frac{1}{2}\left(1(-\frac{4}{3}t) + \frac{4}{3}(t)\right) = 0$

$\int_{C_1}\vec{F}\cdot d\vec{r} = 0$

C_3: $\vec{r}(t) = \langle t, -\frac{4}{3}t\rangle$, $t = -3..0$ (reversed direction)

$\vec{r}'(t) = \langle 1, -4/3\rangle$

$\vec{F}(\vec{r}(t)) = \frac{1}{2}\langle \frac{4}{3}t, t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = \frac{1}{2}\left(1(\frac{4}{3}t) + (-\frac{4}{3})t\right) = 0$

$\int_{C_3}\vec{F}\cdot d\vec{r} = 0$

C_2: $\vec{r}(t) = \langle 5\cos t, 5\sin t\rangle$ $t = \theta_1..\theta_2$

$\vec{r}'(t) = \langle -5\sin t, 5\cos t\rangle$

$\vec{F}(\vec{r}(t)) = \frac{1}{2}\langle -5\sin t, 5\cos t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = \frac{1}{2}\left(-5\sin t(-5\sin t) + 5\cos t(5\cos t)\right)$

$= \frac{25}{2}(\sin^2 t + \cos^2 t) = \frac{25}{2}$

$$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{\theta_1}^{\theta_2}\frac{25}{2}\,dt = \frac{25}{2}\Big|_{\theta_1}^{\theta_2} = \frac{25}{2}(\theta_2 - \theta_1)$$

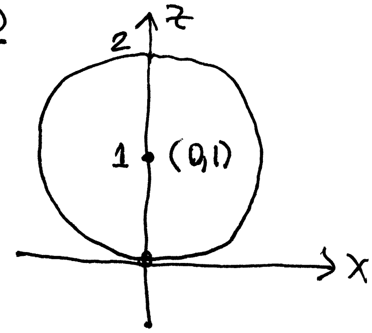
$$= \frac{25}{2}\left(\pi - \arctan 4/3 - \arctan 4/3\right) = \boxed{\frac{25}{2}\pi - 25\arctan 4/3} \boxed{= A}$$

g) $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1$

$$\iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \iint_R 1\, dA = \boxed{A}$$

same ✓
as area integral

MAT 2500-01 13S Final Exam Answers (3)



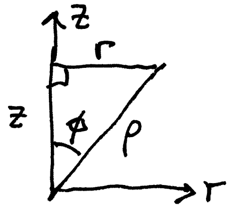
(3) a) $x^2+y^2+(z-1)^2=1$

$$\boxed{r^2+(z-1)^2=1^2}$$

radius 1

center $(r,z)=(0,1)$

b)



$r=\rho\sin\phi$
$z=\rho\cos\phi$

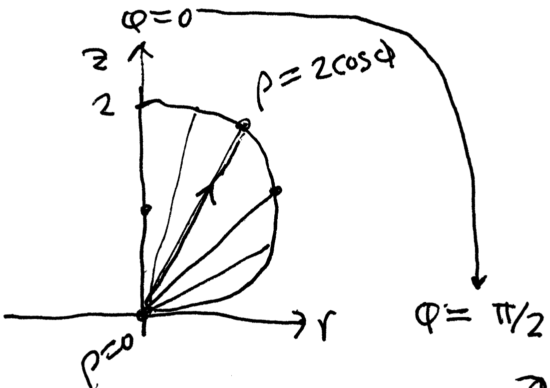
$$(\rho\sin\phi)^2+(\rho\cos\phi-1)^2=1$$

$$\underbrace{\rho^2\sin^2\phi+\rho^2\cos^2\phi}-2\rho\cos\phi+\cancel{1}=\cancel{1}$$

$$\rho^2(\sin^2\phi+\cos^2\phi)$$
$$=\rho^2$$

$$\rho^2=2\rho\cos\phi$$

$$\boxed{\rho=2\cos\phi}$$



c) $\iiint 1\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\underbrace{\int_0^{2\cos\phi} 1\,\underbrace{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}_{dV}}$

$$\underbrace{\frac{\rho^3}{3}\sin\phi\Big|_{\rho=0}^{\rho=2\cos\phi}} = \frac{8}{3}\underbrace{\cos^3\phi}_{u^3}\sin\phi$$

$$\int\frac{8}{3}\underbrace{\cos^3\phi}_{u^3}\,\underbrace{\sin\phi\,d\phi}_{-du} = -\frac{8}{3}\int u^3du = -\frac{8}{3}\frac{u^4}{4} = -\frac{8}{3}\frac{\cos^4\phi}{4}$$

$$= \underbrace{\int_0^{2\pi}d\theta}_{2\pi}\left(-\frac{2}{3}\cos^4\phi\right)\Big|_0^{\frac{\pi}{2}} = 2\pi\left(+\frac{2}{3}\right)\left(-\cos^4\frac{\pi}{2}+\cos^4 0\right) = \boxed{\frac{4\pi}{3}}$$

volume of unit sphere! ✓

$$V_z=\iiint z\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\phi}(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\left(\int_0^{2\pi}d\theta\right)\int_0^{\frac{\pi}{2}}\frac{\rho^4}{4}\Big|_{\rho=0}^{\rho=2\cos\phi}\cos\phi\sin\phi\,d\phi = \frac{2\pi}{4}\int_0^{\frac{\pi}{2}}(16\cos^5\phi)\sin\phi\,d\phi$$

$$=8\pi\left(-\frac{\cos^6\phi}{6}\right)\Big|_{\phi=0}^{\phi=\pi/2} = \frac{4\pi}{3}(-0+1)$$
$$=\frac{4\pi}{3}$$

$$\bar{z}=\frac{V_z}{V}=\frac{4\pi/3}{4\pi/3}=\boxed{1}$$ ✓ of course

MAT2500-01 13S Final Exam Answers

④ a) $\vec{F} = \langle x, -y, z \rangle \stackrel{?}{=} \vec{\nabla} f$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ x & -y & z \end{vmatrix} = \langle \partial_y z - \partial_z(-y), \partial_z x - \partial_x z, \partial_x(-y) - \partial_y(x) \rangle = \vec{0}$$

$$\int \left[\frac{\partial f}{\partial x} = x \right] dx \qquad f = \int x\,dx = \frac{x^2}{2} + C(y,z) \to \frac{\partial f}{\partial y} = 0 + \frac{\partial C}{\partial y}(y,z)$$

$$\frac{\partial f}{\partial y} = -y \quad = \quad \frac{\partial C}{\partial y}(y,z) \longrightarrow C(y,z) = -\int y\,dy$$

$$= -\frac{y^2}{2} + C(z)$$

$$\frac{\partial f}{\partial z} = z$$

$$f = \frac{x^2}{2} - \frac{y^2}{2} + C(z)$$

$$\frac{\partial f}{\partial z} = 0 + C'(z)$$

$$C'(z) = z$$

$$C = \int z\,dz \cancel{+k} \quad \frac{z^2}{2} + k$$

$$f = \frac{x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2} + k$$

$$f(3,1,2) - f(1,2,3) = \frac{1}{2}(3^2 - 1^2 + 2^2) - \frac{1}{2}(1^2 - 2^2 + 3^2)$$

$$= \frac{1}{2}(3^2 - 3^2 + 2^2 + 2^2 - 1 - 1) = +\frac{6}{2} = \boxed{3}$$

optional check: $\vec{r} = \langle 1,2,3 \rangle + t\,(\langle 3,1,2 \rangle - \langle 1,2,3 \rangle)$

$$= \langle 1,2,3 \rangle + t\langle 2,-1,-1 \rangle = \langle 1+2t, 2-t, 3-t \rangle, \quad t = 0..1$$

$$\vec{r}\,' = \langle 2,-1,-1 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle +(1+2t), -(2-t), 3-t \rangle$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 2(1+2t) + (2-t) - (3-t)$$

$$= 2 + 4t + 2 - t - 3 + t = 4t + 1$$

$$\int_C F \cdot dr = \int_0^1 (4t+1)\,dt = 4\left(\frac{t^2}{2}\right) + t \Big|_0^1 = 2 + 1 = \boxed{3} \checkmark$$