MAT2705-01/02 13F Quiz 9 Print Name (Last, First) ____________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). You may use technology to give the RREF form of matrices.

1. $x_1'(t) = 4x_1(t) - 3x_2(t) + x_3(t)$, $x_2'(t) = 2x_1(t) - x_2(t) + x_3(t)$, $x_3'(t) = 2x_3(t)$,
$x_1(0) = 2, x_2(0) = 1, x_3(0) = -3$.

a) Rewrite this system of DEs **and** its initial conditions explicitly in matrix form for the vector variable $\boldsymbol{x} = \langle x_1, x_2, x_3\rangle$ as a column matrix (using the actual matrices, not their symbols), identifying the coefficient matrix A.

b) For this A, using Maple write down the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (ordered by increasing absolute value, they are integers!) and corresponding matrix of eigenvectors $B_{maple} = \langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle$ that it provides you, reordering them if necessary to order them as requested.

c) By hand showing all steps (you should use technology to evaluate the necessary determinant and to solve the resulting condition), show that the characteristic equation (state it explicitly) for the eigenvalues has roots 1 and 2 . What are their respective multiplicities?

d) For each eigenvalue, by hand find a basis of the corresponding eigenspace, collecting your results into a new matrix B and compare your result with Maple's. Do they agree once reordered as above? If not, are they equivalent modulo permutations or rescalings?

e) Use technology to evaluate and write down the inverse matrix B^{-1} and use Maple to evaluate the matrix product $A_D = B^{-1}A\,B$. Write down this result. Does it evaluate correctly to the diagonalized matrix with the eigenvalues in the correct order?

f) Given that $\vec{x} = B\,\vec{y}$, if $\vec{x}(0) = \langle 2, 1, -3\rangle$, find $\vec{y}(0)$. Show how you did this.

g) **Optional.** Finally solve the decoupled equations $\vec{y}\,' = A_D\,\vec{y}$ and write out the solution $\vec{x} = B\,\vec{y}$ with the initial condition $\vec{y}(0)$ imposed on the arbitrary constants, first as a linear combination of the modes (vector form) and then multiplied out to identify the individual components of the vector (scalar form).

▶ **solution**

① a) $$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

b) Maple: (possibly reordered) $$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad B_{maple} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

c) $$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} \overset{\text{Maple}}{=} -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda-1)(\lambda-2)^2 = 0$$

↑ factor

roots: $\lambda = 1, m = 1$
$\lambda = 2, m = 2$ (eigenvalues & multiplicities)

d) $\lambda = 1$: $$A - I = \begin{bmatrix} 4-1 & -3 & 1 \\ 2 & -1-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{c} \text{L F L} \\ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ x_1\ x_2\ x_3 \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = t$
$\begin{cases} x_1 - x_2 = 0 \to x_1 = t \\ x_3 = 0 \end{cases}$ $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}_1}$$

d) $\lambda = 2$: $$A - 2I = \begin{bmatrix} 4-2 & -3 & 1 \\ 2 & -1-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{c} \text{L F F} \\ \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = t_1, x_3 = t_2$
$x_1 - 3/2\,x_2 + \frac{1}{2}x_3 = 0 \to x_1 = \frac{3}{2}t_1 - \frac{1}{2}t_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/2\,t_1 - \frac{1}{2}t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \underbrace{\begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}_2} + t_2 \underbrace{\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{b}_3}.$$

$$B = \begin{bmatrix} 1 & 3/2 & -1/2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ switched compared to my Maple, reordered by eigenvalue.

$$A_B = B^{-1}AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ yes!

↑

$$B^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ easy by hand too!

f) $\vec{x}(0) = B\,\vec{y}(0) \to \vec{y}(0) = B^{-1}\vec{x}(0) \overset{\text{maple}}{=} \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$

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9) $\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ 2y_2 \\ 2y_3 \end{bmatrix}$ $y_1' = y_1$, $y_2' = 2y_2$, $y_3' = 2y_3$ $y_1 = c_1 e^t$, $y_2 = c_2 e^{2t}$, $y_3 = c_3 e^{2t}$ $y_1(0) = c_1 = 2$, $y_2(0) = c_2 = -1$, $y_3(0) = c_3 = -3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{B}\vec{y} = \begin{bmatrix} 1 & 3/2 & -1/2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2e^t \\ -1e^{2t} \\ -3e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^t - \frac{3}{2}e^{2t} + \frac{3}{2}e^{2t} \\ 2e^t - e^{2t} \\ -3e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^t \\ 2e^t - e^{2t} \\ -3e^{2t} \end{bmatrix}$$

$\boxed{x_1 = 2e^t,\ x_2 = 2e^t - e^{2t},\ x_3 = -3e^{2t}}$ scalar soln

oops

$$\left.\begin{aligned} &= 2e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - e^{2t} \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix} - 3e^{2t} \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \\ &= 2e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \underbrace{e^{2t} \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}} \end{aligned}\right\} \text{vector "mode" soln}$$

degenerate modes associated with twice repeated eigenvalue
collapse to particular direction
in span $\{\vec{b}_2, \vec{b}_3\}$ (eigenspace for $\lambda = 2$)