MAT2705-01/02 13F Final Exam Answers (1)

a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \begin{bmatrix} -7/9 & 4/9 \\ 1/3 & -1/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 15\cos t/2 \end{bmatrix} = \begin{bmatrix} -7/9\, x_1 + 4/9\, x_2 \\ 1/3\, x_1 - 1/3\, x_2 + 15\cos t/2 \end{bmatrix}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$$\boxed{\begin{aligned} x_1'' &= -\tfrac{7}{9}x_1 + \tfrac{4}{9}x_2 \\ x_2'' &= \tfrac{1}{3}x_1 - \tfrac{1}{3}x_2 + 15\cos t/2 \\ &x_1(0)=0,\ x_2(0)=0,\ x_1'(0)=0,\ x_2'(0)=4 \end{aligned}} \xrightarrow{\text{Maple}} \boxed{\begin{aligned} x_1 &= -64\cos t/2 + 10\cos t - 2\sin t + 54\cos t/3 + 6\sin t/3 \\ x_2 &= -76\cos t/2 - 5\cos t + \sin t + 81\cos t/3 + 9\sin t/3 \end{aligned}}$$

$\omega = 1/2$ $\qquad$ $\omega_2 = 1$ $\qquad$ $\omega_1 = 1/3$

$T = \frac{2\pi}{1/2} = 4\pi$ $\qquad$ $T_2 = \frac{2\pi}{1} = 2\pi$ $\qquad$ $T_1 = \frac{2\pi}{1/3} = 6\pi$

$12\pi$: 3 cycles $(=3T)$ $\qquad$ 6 cycles $(=6T_2)$ $\qquad$ 2 cycles $(=2T_1)$

b) $0 = |A - \lambda I| = \begin{vmatrix} -\frac{7}{9}-\lambda & \frac{4}{9} \\ \frac{1}{3} & -\frac{1}{3}-\lambda \end{vmatrix} = (\lambda + \tfrac{7}{9})(\lambda + \tfrac{1}{3}) - \tfrac{4}{27} = \lambda^2 + \tfrac{10}{9}\lambda + \underbrace{\tfrac{7}{27} - \tfrac{4}{27}}_{=3/27=1/9}$

factor or just solve with Maple

$= (\lambda + \tfrac{1}{9})(\lambda + 1) \rightarrow \lambda_1 = -1/9,\ \lambda_2 = -1$ $\qquad |\lambda_1| < |\lambda_2| \rightarrow$ $\omega_1 = \sqrt{|-1/9|} = 1/3$

$\omega_2 = \sqrt{|-1|} = 1$

$\lambda_1 = -\frac{1}{9}$ $\quad A + \frac{1}{9}I = \begin{bmatrix} -\frac{7}{9}+\frac{1}{9} & \frac{4}{9} \\ \frac{1}{3} & -\frac{1}{3}+\frac{1}{9} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{4}{9} \\ \frac{1}{3} & -\frac{2}{9} \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

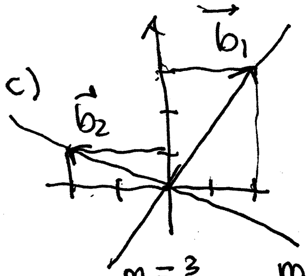
$x_1 - \frac{2}{3}x_2 = 0$, $x_2 = t$, $x_1 = \frac{2}{3}t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2/3\, t \\ t \end{bmatrix} = \frac{t}{3}\underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\vec{b}_1}$

$\lambda_2 = -1$ $\quad A + I = \begin{bmatrix} -\frac{7}{9}+1 & \frac{4}{9} \\ \frac{1}{3} & -\frac{1}{3}+1 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{4}{9} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 = 0$, $x_2 = t$, $x_1 = -2t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

$B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$, $B^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$ $\quad A_B = B^{-1}AB = \begin{bmatrix} -1/9 & 0 \\ 0 & -1 \end{bmatrix}$

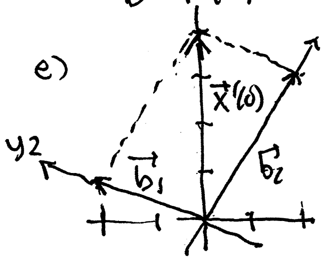
c)




d) $B^{-1}\vec{x}'(0) = \frac{1}{8}\begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$B^{-1}\vec{F}(0) = \frac{1}{8}\begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 15 \end{bmatrix} = \begin{bmatrix} 15/4 \\ 15/4 \end{bmatrix}$

e)




$\vec{x}'(0) = \vec{b}_1 + \vec{b}_2$

# MAT 2705-01/02 Final Exam Answers (2)

f) $\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -1/9 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 15/4 \cos t/2 \\ 15/4 \cos t/2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9}y_1 + \frac{15}{4}\cos\frac{t}{2} \\ -y_2 + \frac{15}{4}\cos\frac{t}{2} \end{bmatrix}$

$y_1'' + \frac{1}{9}y_1 = \frac{15}{4}\cos\frac{t}{2} \qquad y_{1h} = c_1\cos\frac{t}{3} + c_2\sin\frac{t}{3} \qquad y_{1p} = c_5\cos\frac{t}{2} + c_6\sin\frac{t}{2}$

$y_2'' + y_1 = \frac{15}{4}\cos\frac{t}{2} \qquad y_{2h} = c_3\cos t + c_4\sin t \qquad y_{2p} = c_7\cos\frac{t}{2} + c_8\sin\frac{t}{2}$

$\frac{1}{9}\left[\, y_{1p} = c_5\cos\frac{t}{2} + c_6\sin\frac{t}{2} \right]$

$1\left[\, y_{1p}'' = -\frac{1}{4}c_5\cos\frac{t}{2} - \frac{1}{4}c_6\sin\frac{t}{2} \right]$

$y_{1p}'' + \frac{1}{9}y_{1p} = \underbrace{\left(\frac{1}{9} - \frac{1}{4}\right)}_{-5/36} c_5\cos\frac{t}{2} + \underbrace{\left(\frac{1}{9} - \frac{1}{4}\right)}_{-5/36} c_6\sin\frac{t}{2} = \frac{15}{4}\cos\frac{t}{2} + 0\sin\frac{t}{2} \qquad -\frac{5}{36}c_5 = \frac{15}{4} \quad c_5 = -27$

$-\frac{5}{36}c_6 = 0 \quad c_6 = 0$

$1\left[\, y_{2p}'' = c_7\cos\frac{t}{2} + c_8\sin\frac{t}{2} \right]$

$1\left[\, y_{2p}'' = -\frac{1}{4}c_7\cos t - \frac{1}{4}c_8\sin t \right]$

$y_{2p}'' + y_{2p} = \underbrace{\left(1 - \frac{1}{4}\right)}_{3/4} c_7\cos\frac{t}{2} + \underbrace{\left(1 - \frac{1}{4}\right)}_{3/4} c_8\sin\frac{t}{2} = \frac{15}{4}\cos\frac{t}{2} + 0\sin\frac{t}{2} \qquad \frac{3}{4}c_7 = \frac{15}{4} \cdot c_7 = 5$

$\frac{3}{4}c_8 = 0 \quad c_8 = 0$

$y_{1p} = -27\cos\frac{t}{2}, \quad y_{2p} = 5\cos\frac{t}{2}$

$$\boxed{\begin{aligned} y_1 &= \underbrace{c_1\cos\tfrac{t}{3} + c_2\sin\tfrac{t}{3}}_{y_{ih}} \underbrace{-27\cos\tfrac{t}{2}}_{y_{ip}} \\ y_2 &= c_3\cos t + c_4\sin t + 5\cos\tfrac{t}{2} \end{aligned}} \longrightarrow \begin{aligned} y_{1h} &= 27\cos\tfrac{t}{3} + 3\sin\tfrac{t}{3} \\ y_{2h} &= -5\cos t + \sin t \end{aligned}$$

g) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c_1\cos\frac{t}{3} + c_2\sin\frac{t}{3} - 27\cos\frac{t}{2} \\ c_3\cos t + c_4\sin t + 5\cos\frac{t}{2} \end{bmatrix}$

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3}c_1\sin\frac{t}{3} + \frac{1}{3}c_2\cos\frac{t}{3} + \frac{27}{2}\sin\frac{t}{2} \\ -c_3\sin t + c_4\cos t - \frac{5}{2}\sin\frac{t}{2} \end{bmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c_1 - 27 \\ c_3 + 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 - 27 \\ c_3 + 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{aligned} c_1 &= 27 \\ c_3 &= -5 \end{aligned}$

$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3}c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} c_2/3 \\ c_4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{aligned} c_2 &= 3 \\ c_4 &= 1 \end{aligned}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 27\cos\frac{t}{3} + 3\sin\frac{t}{3} - 27\cos\frac{t}{2} \\ -5\cos t + \sin t + 5\cos\frac{t}{2} \end{bmatrix}$

$= \begin{bmatrix} (54\cos\frac{t}{3} + 6\sin\frac{t}{3} - 54\cos\frac{t}{2}) - (-10\cos t + 2\sin t + 10\cos\frac{t}{2}) \\ (81\cos\frac{t}{3} + 9\sin\frac{t}{3} - 81\cos\frac{t}{2}) + (-5\cos t + \sin t + 5\cos\frac{t}{2}) \end{bmatrix}$

$= \begin{bmatrix} 54\cos\frac{t}{3} + 6\sin\frac{t}{3} + 10\cos t - 2\sin t - 64\cos t/2 \\ 81\cos\frac{t}{3} + 9\sin\frac{t}{3} - 5\cos t + \sin t - 76\cos t/2 \end{bmatrix}$ yeah! same as Maple!

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g) continued:
$$\boxed{\begin{aligned} x_1 &= 54\cos\tfrac{t}{3} + 6\sin\tfrac{t}{3} + 10\cos t - 2\sin t - 64\cos t/2 \\ x_2 &= 81\cos\tfrac{t}{3} + 9\sin\tfrac{t}{3} - 5\cos t + \sin t - 76\cos t/2 \end{aligned}}$$

Just re-ordered compared to Maple.

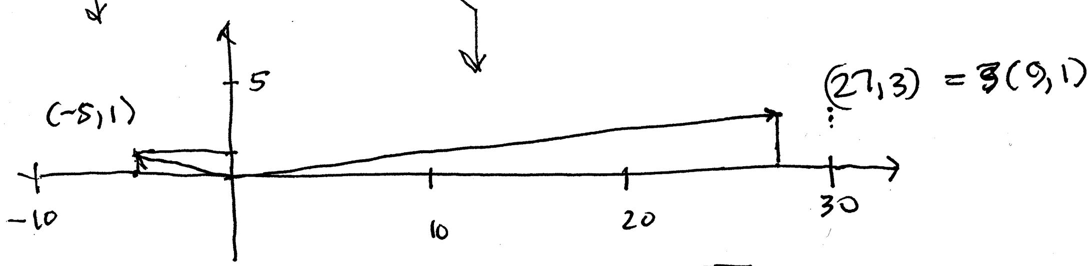
h) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} y_{1h} \\ y_{2h} \end{bmatrix} + \begin{bmatrix} -64\cos t/2 \\ -76\cos t/2 \end{bmatrix}$$

see $y_{1h}, y_{2h}$ above

$$= \underbrace{\left(27\cos\tfrac{t}{3} + 3\sin\tfrac{t}{3}\right)}_{y_{1h}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \underbrace{(-5\cos t + \sin t)}_{y_{2h}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \cos\tfrac{t}{2} \underbrace{\begin{bmatrix} -64 \\ -76 \end{bmatrix}}_{\vec{b}_3}$$

$(27,3) = 3(9,1)$ $\quad$ $(-5,1)$

tandem mode $\quad$ accordian mode




i)

$A_2 = \sqrt{5^2+1} = \sqrt{26}$
$\approx 5.099$

$\delta_2 = \pi - \arctan 1/5$
($\approx 0.466$ cycles)
$\approx 168.7°$

$A_1 = 3\sqrt{9^2+1} = 3\sqrt{82}$
$\approx 27.166$

$\delta_1 = \arctan 1/9$
($\approx 0.018$ cycles)
($\approx 6.3°$) ← not requested.

$\cancel{\boxed{y_{1h} = \sqrt{26}\cos\left(\tfrac{t}{3} - \pi + \arctan\tfrac{1}{5}\right)}}$ $\quad$ $\cancel{\boxed{y_{2h} = 3\sqrt{82}\cos\left(t - \arctan 1/9\right)}}$

oops, exchanged variables in backsubstitution because of left, right zigzag!
this is an obvious typo that should be clear from context!

$\boxed{y_{2h} = \sqrt{26}\cos\left(t - \pi + \arctan 1/5\right)}$ $\quad$ $\boxed{y_{1h} = 3\sqrt{82}\cos\left(\tfrac{t}{3} - \arctan 1/9\right)}$

if something seems wrong in an answer key, THINK IT OUT: have some confidence in what you have learned. And report such glitches to bob please.