MAT 2705-01/02 13F Takehome Test 3 Answers (1).

(1) a) $2x'' + 8x' + 136x = F$

$\tfrac12 \curvearrowright$ $\boxed{x'' + 4x' + 68x = F/2}$

$k_0 = 4$, $\tau_0 = k_0^{-1} = 1/4 = 0.25$

$\omega_0 = \sqrt{68} \approx 8.246$

$Q = \omega_0 \tau_0 = \frac14 2\sqrt{17} = \frac{\sqrt{17}}{2} \approx 2.062$

$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\sqrt{17}} = \frac{\pi}{\sqrt{17}} \approx 0.762$

b) $x = e^{rt}$: $(r^2+4r+68)e^{rt} = 0$

$r^2+4r+68 = 0 \to r = -2 \pm 8i$

$x = e^{(-2\pm 8i)t} = e^{-2t}(\cos 8t \pm i \sin 8t)$

$\hookrightarrow e^{-2t}\cos 8t,\ e^{-2t}\sin 8t$

gen soln:

$\boxed{x = e^{-2t}(c_1 \cos 8t + c_2 \sin 8t)}$

$\omega = 8$ $\quad \tau_1 = 1/2 = 0.5$

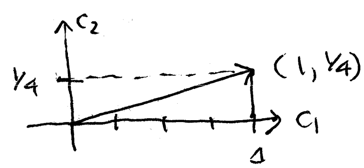
$T_1 = 2\pi/8 = \pi/4 \approx 0.785$

c) $x' = -2e^{-2t}(c_1\cos 8t + c_2 \sin 8t)$
$\quad + e^{-2t}(-8c_1 \sin 8t + 8c_2 \cos 8t)$

$x(0) = c_1 = 1$

$x'(0) = -2c_1 + 8c_2 = 0 \to c_2 = \frac{2}{8}(1) = \frac14$

$\boxed{x = e^{-2t}(\cos 8t + \tfrac14 \sin 8t)}$



$A = \sqrt{1+\frac{1}{16}} = \sqrt{\frac{17}{16}} = \frac{\sqrt{17}}{4} \approx 1.031$

$\delta = \arctan 1/4$

$x = \frac{\sqrt{17}}{4} e^{-2t} \cos(8t - \arctan 1/4)$

$\underbrace{\frac{\sqrt{17}}{4} e^{-2t}}_{}$

$A(t) = \frac{\sqrt{17}}{4} e^{-2t}$ $\quad \approx 1.031$

envelope functions: $\boxed{\pm \frac{\sqrt{17}}{4} e^{-2t}}$

[Note $\frac{5\tau_1}{T_1} \approx 3.18 \rightsquigarrow$ 3 oscillations in decay window]

d) $F/2 = 65(\cos 8t + \sin 8t)$

$68\ [x_p = c_3 \cos 8t + c_4 \sin 8t]$

$4\ [x_p' = -8c_3 \sin 8t + 8c_4 \cos 8t]$

$1\ [x_p'' = -64c_3 \cos 8t - 64 c_4 \sin 8t]$

$x_p'' + 4x_p' + 68x_p = [(68-64)c_3 + 4\cdot 8 c_4]\cos 8t$
$\quad + [-4\cdot 8c_3 + (68-64)c_4]\sin 8t$
$\quad = 65\cos 8t + 65 \sin 8t$

$4c_3 + 32c_4 = 65$
$-32c_3 + 4c_4 = 65$

$\begin{bmatrix} 4 & 32 \\ -32 & 4\end{bmatrix}\begin{bmatrix} c_3 \\ c_4\end{bmatrix} = \begin{bmatrix} 65 \\ 65 \end{bmatrix}$

$\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \left(4\begin{bmatrix} 1 & 8 \\ -8 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix} 65 \\ 65\end{bmatrix} = \frac14 \frac{1}{1+64}\begin{bmatrix} 1 & -8 \\ 8 & 1\end{bmatrix}\begin{bmatrix} 65 \\ 65\end{bmatrix}$

$= \frac14 \frac{65}{65}\begin{bmatrix} 1 \\ 8 \end{bmatrix} = \begin{bmatrix} -7/4 \\ 9/4\end{bmatrix}$

$\boxed{x_p = \tfrac14(-7\cos 8t + 9 \sin 8t)}$

$x = x_h + x_p = e^{-2t}(c_1 \cos 8t + c_2 \sin 8t) + \frac14(-7\cos 8t + 9 \sin 8t)$

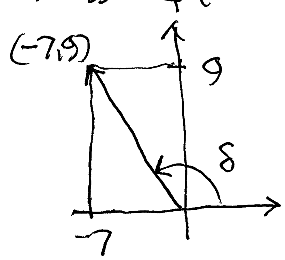
$x' = -2e^{-2t}(c_1 \cos 8t + c_2 \sin 8t) + \frac14(56 \sin 8t + 72 \cos 8t)$
$\quad + e^{-2t}(-8c_1 \sin 8t + 8c_2 \cos 8t)$

$x(0) = c_1 - 7/4 = 0 \to c_1 = 7/4$

$x'(0) = -2c_1 + 8c_2 + \frac{72}{4} = 0 \to c_2 = \frac18\left(2\left(\frac74\right) - \frac{72}{4}\right)$
$= -29/16$

$\boxed{x = \left(-\frac{29}{16}\sin 8t + \frac74 \cos 8t\right)e^{-2t} \underbrace{- \frac74 \cos 8t + \frac94 \sin 8t}_{x_{ss}\ \text{steady state soln}}}$

e) $x_{ss} = \frac14(-7\cos 8t + 9 \sin 8t)$



$A = \frac14\sqrt{(-7)^2 + 9^2} = \frac14 \sqrt{49+81} = \boxed{\frac14\sqrt{130} \approx 2.850}$

$\delta \approx \pi - \arctan\frac97 \approx 2.232 \approx 127.9° \approx 0.355$ cycles

$\delta - \delta_s = \pi - \arctan\frac97 - \frac{\pi}{4}$
$= \frac{3\pi}{4} - \arctan\frac97 \approx$ 1.446 radians $\approx$ 0.230 cycles $\approx 82.9°$

so $x_{ss} = \frac{\sqrt{130}}{4}\cos(8t - \pi + \arctan 9/7)$

f) shifted about 1/4 cycle to right compared to driving function (later in time)

f) plot $\frac{\sqrt{130}}{4\sqrt2}(\cos 8t + \sin 8t)$, $\frac14(-7\cos 8t + 9 \sin 8t)$

one period $T_1 \approx 0.785$. $(= 2\pi/8 = \pi/4)$

exponential decay window $5\tau_1 = 2.5$ for decay plot.

MAT 2705-01/02 13F Takehome Test 3 Answers (2)

(1) g) $x'' + 4x' + 68x = \frac{F_0}{2}(\cos\omega t + \sin\omega t)$

$$
\begin{aligned}
68\ [x_p &= c_3\cos\omega t + c_4\sin\omega t]\\
4\ [x_p' &= -\omega c_3\sin\omega t + \omega c_4\cos\omega t]\\
1\ [x_p'' &= -\omega^2 c_3\cos\omega t - \omega^2 c_4\sin\omega t]
\end{aligned}
$$

$$
\begin{aligned}
x_p'' + 4x_p' + 68x_p &= [(68-\omega^2)c_3 + 4\omega c_4]\cos\omega t\\
&\quad + [-4\omega c_3 + (68-\omega^2)c_4]\sin\omega t\\
&= \frac{F_0}{2}\cos\omega t + \frac{F_0}{2}\sin\omega t
\end{aligned}
$$

$$
\begin{aligned}
(68-\omega^2)c_3 + 4\omega c_4 &= F_0/2\\
-4\omega c_3 + (68-\omega^2)c_4 &= F_0/2
\end{aligned}
$$

$$\begin{bmatrix}(68-\omega^2) & 4\omega\\ -4\omega & (68-\omega^2)\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}F_0/2\\ F_0/2\end{bmatrix} = \frac{F_0}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}\downarrow\end{bmatrix}^{-1}\begin{bmatrix}F_0/2\\ F_0/2\end{bmatrix} = \frac{F_0/2}{(68-\omega^2)^2+16\omega^2}\begin{bmatrix}68-\omega^2 & -4\omega\\ 4\omega & 68-\omega^2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{F_0}{2\,[(68-\omega^2)^2+16\omega^2]}\begin{bmatrix}68-\omega^2-4\omega\\ 68-\omega^2+4\omega\end{bmatrix}$$

$$\boxed{x_p = \frac{F_0/2}{(68-\omega^2)^2+16\omega^2}\left[(68-\omega^2-4\omega)\cos\omega t + (68-\omega^2+4\omega)\sin\omega t\right]}$$

(steady state soln)

h) $$A(\omega) = \frac{F_0/2}{(68-\omega^2)^2+16\omega^2}\left((68-\omega^2-4\omega)^2 + (68-\omega^2+4\omega)^2\right)^{1/2}$$

$$\underset{\text{maple}}{\overset{\text{expand}}{=}} \frac{F_0/2\,[2(\omega^4-120\omega^2+4624)]^{1/2}}{(\omega^4-120\omega^2+4624)}$$

$$= \frac{F_0}{\sqrt2}(\omega^4-120\omega^2+4624)^{-1/2}$$

$$\boxed{A(\omega) = F_0\,(2\omega^4-240\omega^2+9248)^{-1/2}}$$

$$A'(\omega) = F_0(-\tfrac12)(\ldots)^{-3/2}\underbrace{(8\omega^3-480\omega)}_{8\omega(\omega^2-60)} = 0$$

$\omega = 0, \sqrt{60}$ $\quad \boxed{\omega_p = \sqrt{60} = 2\sqrt{15} \approx 7.746}$

$$A(\omega_p) \overset{\text{maple}}{=} \frac{65}{32}\sqrt2 \approx 2.873$$

$$A(8) = \tfrac14\sqrt{130} \approx 2.850 \quad (\text{if } F_0 = 130)$$

$$A(0) = \frac{65\sqrt{2\cdot 68^2}}{68^2} = \frac{65}{68}\sqrt2 \approx 1.352$$

h) $A(\omega_p)/A(0) \underset{\text{maple}}{=} \frac{17}{8} = 2.125 \Leftrightarrow Q \approx 2.061$
pretty close

i) Window $\omega = 0..30$ seems to show the peak and tail well.

(2) a) $(k_1, k_2, k_3) = \left(\frac66, \frac62, \frac66\right) = (1, 3, 1)$

$$\begin{bmatrix}x_1'\\ x_2'\\ x_3'\end{bmatrix} = \begin{bmatrix}-x_1 + 0x_2 + x_3\\ x_1 - 3x_2 + 0x_3\\ 0x_1 + 3x_2 - x_3\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}' = \underbrace{\begin{bmatrix}-1 & 0 & 1\\ 1 & -3 & 0\\ 0 & 3 & -1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \qquad \begin{bmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix} = \begin{bmatrix}14\\0\\0\end{bmatrix}$$

b) $$0 = |A-\lambda I| = \begin{vmatrix}-1-\lambda & 0 & 1\\ 1 & -3-\lambda & 0\\ 0 & 3 & -1-\lambda\end{vmatrix}$$

Maple $-(\lambda^3+5\lambda^2+7\lambda) = -\lambda(\lambda^2+5\lambda+7)$

$\hookrightarrow \lambda = 0,\ -\frac52 \pm \frac{i\sqrt3}{2}$

$$\lambda = 0:\ A = \begin{bmatrix}-1 & 0 & 1\\ 1 & -3 & 0\\ 0 & 3 & -1\end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{c}\begin{array}{ccc}L & L & F\end{array}\\ \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1/3\\ 0 & 0 & 0\end{bmatrix}\end{array}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_3 = t \quad x_1 - x_3 = 0 \to x_1 = t$
$\qquad\quad x_2 - x_3/3 = 0 \to x_2 = t/3$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}t\\ t/3\\ t\end{bmatrix} = t\underbrace{\begin{bmatrix}1\\ 1/3\\ 1\end{bmatrix}}_{\vec b_1}$$

$$A - \left(-\tfrac52 + \tfrac{i\sqrt3}{2}\right)I = \begin{bmatrix}-1+\frac52-\frac{i\sqrt3}{2} & 0 & 1\\ 1 & -3+\frac52-\frac{i\sqrt3}{2} & 0\\ 0 & 3 & -1+\frac52-\frac{i\sqrt3}{2}\end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{array}{c}\begin{bmatrix}1 & 0 & \frac12+\frac{i\sqrt3}{6}\\ 0 & 1 & \frac12 - i\sqrt3/6\\ 0 & 0 & 0\end{bmatrix}\\ \begin{array}{ccc}L & L & F\end{array}\end{array}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_3 = t \quad x_1 + \left(\frac12 + \frac{i\sqrt3}{6}\right)x_3 = 0 \to x_1 = -\left(\frac12 + \frac{\sqrt3}{6}i\right)t$
$\qquad\quad x_2 + \left(\frac12 - \frac{i\sqrt3}{6}\right)x_3 = 0 \to x_2 = -\left(\frac12 - \frac{i\sqrt3}{6}\right)t$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}-(1/2+\sqrt3 i/6)\,t\\ -(1/2 - i\sqrt3/6)\,t\\ t\end{bmatrix} = t\begin{bmatrix}-1/2-\sqrt3 i/6\\ -1/2+\sqrt3 i/6\\ 1\end{bmatrix} \leftarrow \text{simplest form}$$

$$\vec b_2 = \begin{bmatrix}2/(-3+\sqrt3 i)\\ -2/(3+\sqrt3 i)\\ 1\end{bmatrix}$$

unrationalized Mapleform $\longrightarrow$

$\vec b_3 = \overline{\vec b_2}$

MAT 2705-01/02 13F Take Home Test 3 Answers (3)

(2) b) continued

$$B = \begin{bmatrix} 1 & -1/2-\sqrt{3}i/6 & -1/2+\sqrt{3}i/6 \\ 1/3 & -1/2+\sqrt{3}i/6 & -1/2-\sqrt{3}i/6 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\lambda = \quad 0 \qquad -\tfrac{5}{2}+\tfrac{\sqrt{3}}{2}i \qquad -\tfrac{5}{2}-\tfrac{\sqrt{3}}{2}i$$

$$\vec{x} = B\vec{y} \qquad \vec{y} = B^{-1}\vec{x}$$

$$\vec{x}\,' = Ax \to (B\vec{y})' = A(B\vec{y})$$

$$B^{-1}(B\vec{y}\,') = \underbrace{B^{-1}AB}_{=A_B}\vec{y}$$

$$\vec{y}\,' = A_B\vec{y}\,, \quad A_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5/2+\sqrt{3}/2\,i & 0 \\ 0 & 0 & -5/2-\sqrt{3}/2\,i \end{bmatrix}$$

$$y_1' = 0 \qquad\qquad y_1 = c_1$$

$$y_2' = \left(-\tfrac{5}{2}+\tfrac{\sqrt{3}}{2}i\right)y_2 \qquad y_2 = c_2 e^{\left(-\frac{5}{2}+\frac{\sqrt{3}}{2}i\right)t}$$

$$y_3' = \left(-\tfrac{5}{2}-\tfrac{\sqrt{3}}{2}i\right)y_3 \qquad y_3 = c_3 e^{\left(-\frac{5}{2}-\frac{\sqrt{3}}{2}i\right)t}$$

$c_3 = \bar{c}_2$ for $\vec{x}$ real

$$\vec{x} = c_1\begin{bmatrix}1\\1/3\\1\end{bmatrix} + \underbrace{c_2 e^{-5t/2} e^{\sqrt{3}i/2\,t}\begin{bmatrix}-1/2-\sqrt{3}i/6\\-1/2+\sqrt{3}i/6\\1\end{bmatrix}} + \text{c.c.}$$

$$= e^{-\frac{5t}{2}}\left(\cos\tfrac{\sqrt{3}}{2}t + i\sin\tfrac{\sqrt{3}}{2}t\right)\begin{bmatrix}-\frac{1}{2}-\frac{\sqrt{3}}{6}i\\-\frac{1}{2}+\frac{\sqrt{3}}{6}i\\1\end{bmatrix}$$

$$= e^{-5t/2}\begin{bmatrix} -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t + i\left(-\frac{1}{2}\sin\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{6}\cos\frac{\sqrt{3}}{2}t\right) \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t + i\left(-\frac{1}{2}\sin\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{6}\cos\frac{\sqrt{3}}{2}t\right) \\ \cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t \end{bmatrix}$$

$$= e^{-5t/2}\begin{bmatrix} -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t \\ \cos\frac{\sqrt{3}}{2}t \end{bmatrix} + i\, e^{-5t/2}\begin{bmatrix} -\frac{1}{2}\sin\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{6}\cos\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\sin\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{6}\cos\frac{\sqrt{3}}{2}t \\ \sin\frac{\sqrt{3}}{2}t \end{bmatrix}$$

so gen. soln:

$$\boxed{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = c_1\begin{bmatrix}1\\1/3\\1\end{bmatrix} + c_2 e^{-5t/2}\begin{bmatrix} -\frac{1}{2}\cos\frac{\sqrt{3}t}{2} + \frac{\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t \\ \cos\frac{\sqrt{3}}{2}t \end{bmatrix} + c_3 e^{-\frac{5t}{2}}\begin{bmatrix} -\frac{1}{2}\sin\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{6}\cos\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\sin\frac{\sqrt{3}t}{2} + \frac{\sqrt{3}}{6}\cos\frac{\sqrt{3}}{2}t \\ \sin\frac{\sqrt{3}}{2}t \end{bmatrix}}$$

c)

$$\begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{bmatrix} = c_1\begin{bmatrix}1\\1/3\\1\end{bmatrix} + c_2\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix} + c_3\begin{bmatrix}-\sqrt{3}/6\\\sqrt{3}/6\\0\end{bmatrix} = \begin{bmatrix}14\\0\\0\end{bmatrix} \qquad \begin{bmatrix} 1 & -1/2 & -\sqrt{3}/6 \\ 1/3 & -1/2 & \sqrt{3}/6 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}14\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}\cdots\end{bmatrix}^{-1}\begin{bmatrix}14\\0\\0\end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & 4 \\ -5\sqrt{3} & 9\sqrt{3} & 2\sqrt{3} \end{bmatrix}\begin{bmatrix}14\\0\\0\end{bmatrix} = 2\begin{bmatrix}3\\-3\\-5\sqrt{3}\end{bmatrix} = \begin{bmatrix}6\\-6\\-10\sqrt{3}\end{bmatrix}$$

c) continued

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 6\begin{bmatrix}1\\1/3\\1\end{bmatrix} - 6\begin{bmatrix} -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t \\ \cos\frac{\sqrt{3}}{2}t \end{bmatrix} e^{-5t/2}$$

$$-10\sqrt{3}\begin{bmatrix} -\frac{1}{2}\sin\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{6}\cos\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\sin\frac{\sqrt{3}t}{2} + \frac{\sqrt{3}}{6}\cos\frac{\sqrt{3}}{2}t \\ \sin\frac{\sqrt{3}}{2}t \end{bmatrix} e^{-5t/2}$$

$$= \begin{bmatrix}6\\2\\6\end{bmatrix} + e^{-5t/2}\begin{bmatrix} (3+5)\cos\frac{\sqrt{3}}{2}t + (-1+5)\sqrt{3}\sin\frac{\sqrt{3}}{2}t \\ (3-5)\cos\frac{\sqrt{3}}{2}t + (-1-5)\sqrt{3}\sin\frac{\sqrt{3}}{2}t \\ -6\cos\frac{\sqrt{3}}{2}t - 10\sqrt{3}\sin\frac{\sqrt{3}}{2}t \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 6 + e^{-\frac{5t}{2}}\left[8\cos\frac{\sqrt{3}}{2}t + 4\sqrt{3}\sin\frac{\sqrt{3}}{2}t\right] \\ 2 + e^{-\frac{5t}{2}}\left[-2\cos\frac{\sqrt{3}}{2}t - 6\sqrt{3}\sin\frac{\sqrt{3}}{2}t\right] \\ 6 + e^{-\frac{5t}{2}}\left[-6\cos\frac{\sqrt{3}}{2}t - 10\sqrt{3}\sin\frac{\sqrt{3}}{2}t\right] \end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}$$

d) $\vec{x}_\infty = \lim_{t\to\infty}\vec{x} = \begin{bmatrix}6\\2\\6\end{bmatrix}$

e) $\tau = \frac{2}{5}$, $5\tau = 2 \to 3$

choose $t = 0..3$ to see soln pixels merge with horizontal asymptotes

MAT 2705-01/02 13F Takehome Test 3 Answers (4)

③ a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \frac{1}{7}\begin{bmatrix} -20x_1+3x_2 \\ 2x_1-15x_2 \end{bmatrix} = \underbrace{\frac{1}{7}\begin{bmatrix} -20 & 3 \\ 2 & -15 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$0 = |A-\lambda I| = \begin{vmatrix} -\frac{20}{7}-\lambda & \frac{3}{7} \\ \frac{2}{7} & -\frac{15}{7}-\lambda \end{vmatrix} = +\left(\frac{20}{7}+\lambda\right)\left(\frac{15}{7}+\lambda\right) - \frac{6}{49}$

$= \lambda^2 + \frac{35}{7}\lambda + \frac{300-6}{49} = \lambda^2 + 5\lambda + \frac{294}{49}$

$= \lambda^2+5\lambda+6 = (\lambda+2)(\lambda+3)$

$\hookrightarrow \lambda = -2, -3$ note $-2 > -3$

$A+2I = \begin{bmatrix} -\frac{20}{7}+2 & \frac{3}{7} \\ \frac{2}{7} & -\frac{15}{7}+2 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t$ $x_1 - \frac{1}{2}x_2 = 0 \to x_1 = t/2$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/2 \\ t \end{bmatrix} = \frac{t}{2}\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{b}_1}$ (integer entries)

$A+3I = \begin{bmatrix} -\frac{20}{7}+3 & \frac{3}{7} \\ \frac{2}{7} & -\frac{15}{7}+3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{6}{7} \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t$ $x_1 + 3x_2 = 0 \to x_1 = -3t$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} -3 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

$$\begin{array}{ccc} \lambda = & -2 & -3 \end{array}$$
$B = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$ $\vec{x} = B\vec{y}$, $\vec{y} = B^{-1}\vec{x}$

b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \end{bmatrix} \to \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{1+6}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -7 \\ 7 \end{bmatrix} = \frac{1}{7}(7)\begin{bmatrix} -1+3 \\ 2+1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}$

c) $\vec{x}' = A\vec{x} \longrightarrow \vec{y}' = A_B\vec{y}$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2y_1 \\ -3y_2 \end{bmatrix}$ $\begin{array}{l} y_1' = -2y_1 \\ y_2' = -3y_2 \end{array}$ $\begin{array}{l} y_1 = c_1e^{-2t} \\ y_2 = c_2e^{-3t} \end{array}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{-2t} \\ c_2e^{-3t} \end{bmatrix}$ $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3e^{-3t}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2e^{-2t}-9e^{-3t} \\ 4e^{-2t}+3e^{-3t} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$ IVP soln

d) $\vec{x}(0) = 2\vec{b}_1 + 3\vec{b}_2$

arrows of direction field line up along new axes, both point towards origin

e) $\tau_1 = \frac{1}{2}$, $\tau_2 = \frac{1}{3}$

$5\tau_1 = \frac{5}{2} = 2.5$

$t = 0..2.5$ natural decay window

$x_1 + x_2 = 6e^{-2t} - 6e^{-3t}$

$0 = x_1' + x_2' = -12e^{-2t} + 18e^{-3t} \;]\; e^{3t}/6$

$-2e^{t} + 3 = 0$

$e^t = 3/2$ $t = \ln(3/2)$

$\hookrightarrow x_1 + x_2 = 6\left[e^{(\ln\frac{3}{2})(-2)} - e^{(\ln\frac{3}{2})(-3)}\right]$

$= 6\left[\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3\right]$

$= 6\left(\frac{2}{3}\right)^2\left(1-\frac{2}{3}\right) = \frac{8}{9}$

$\boxed{\text{max at } \left(\ln\frac{3}{2}, \frac{8}{9}\right) \approx (0.405, 0.889)}$

For all plots see Maple worksheet