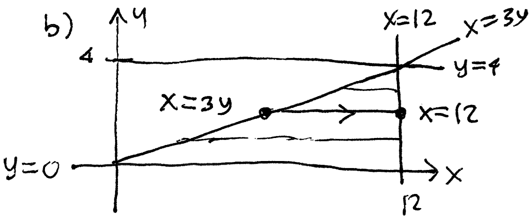
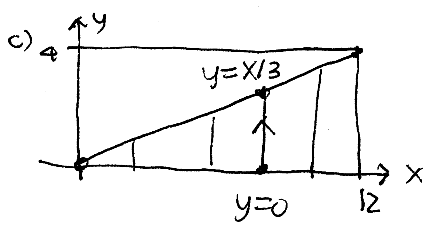
MAT2500-01/04 12S TEST3 Takehome : Answers (1)

① a) $G = \underbrace{\int_{y=0}^{y=4}\int_{x=3y}^{x=12}}\, 6e^{x^2}\,dx\,dy \overset{\text{Maple}}{=} e^{144}-1 \approx 3.4547\times 10^{62}$

b)



c)



d) $\int_0^{12}\int_0^{x/3} 6e^{x^2}\,dy\,dx$

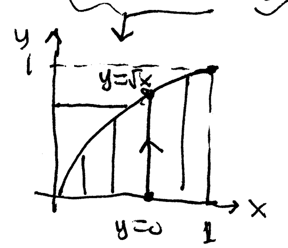
e) $= \int_0^{12} 6e^{x^2} y\Big|_{y=0}^{y=x/3} dx$

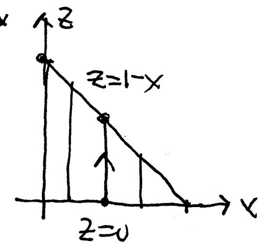
$= \int_0^{12} 6e^{x^2}\left(\frac{x}{3}\right) dx = \int_0^{12} \underbrace{e^{x^2}}_{e^u} \underbrace{(2x\,dx)}_{du}$

$= e^{x^2}\Big|_0^{12} = \boxed{e^{144}-1}$ Maple agrees

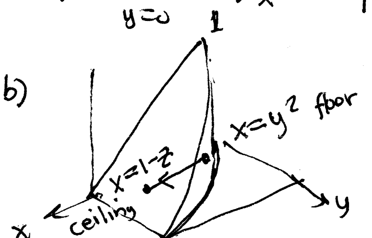
g) gross = dozen dozen (ci) = "unattractively large or bloated" puns require multiple meanings

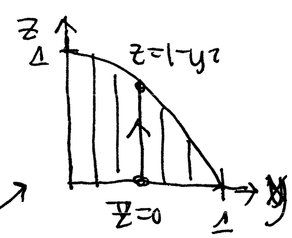
② a) $\underbrace{\int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{x}}}\ \underbrace{\int_{z=0}^{z=1-x}} f(xyz)\,dz\,dy\,dx$





b)

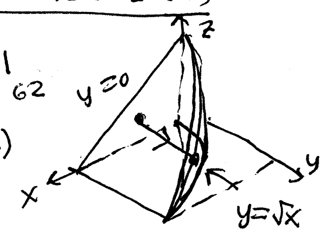


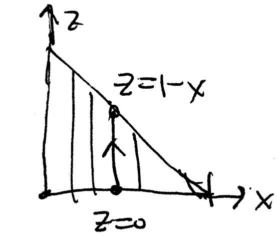


$\begin{cases} x = 1-z \\ x = y^2 \end{cases} \to z = 1-y^2$ intersection

$\boxed{\int_0^1\int_0^{1-y^2}\int_{y^2}^{1-z} f(xyz)\,dx\,dz\,dy}$

②c)





$\boxed{\int_0^1\int_0^{1-x}\int_0^{\sqrt{x}} f(xyz)\,dy\,dz\,dx}$

d) $\int_0^1\int_0^{\sqrt{x}}\underbrace{\int_0^{1-x} 1\,dz}_{z\big|_{z=0}^{z=1-x} = 1-x}\,dy\,dx = \int_0^1\underbrace{\int_0^{\sqrt{x}}(1-x)\,dy}_{(1-x)y\big|_{y=0}^{y=x^{1/2}} = (1-x)x^{1/2}}\,dx$

$= \int_0^1 (1-x)x^{1/2}dx = \int_0^1 x^{1/2} - x^{3/2}\,dx = \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\Big|_0^1$

$= \frac{2}{3} - \frac{2}{5} = 2\left(\frac{5-3}{15}\right) = \boxed{\frac{4}{15}}$

$\int_0^1\int_0^{1-y^2}\underbrace{\int_{y^2}^{1-z} 1\,dx}_{x\big|_{x=y^2}^{x=1-z} = 1-z-y^2}\,dz\,dy = \int_0^1\underbrace{\int_0^{1-y^2} 1-z-y^2\,dz}_{z(1-y^2)-z^2/2\big|_{z=0}^{z=1-y^2}}\,dy$

$= \int_0^1 (1-y^2)^2 - \frac{(1-y^2)^2}{2}\,dy = \frac{1}{2}\int_0^1 (1-y^2)^2 dy = \frac{1}{2}\int_0^1 (1-2y^2+y^4)\,dy$

$= \frac{1}{2}\left(y - \frac{2}{3}y^3 + \frac{y^5}{5}\right)\Big|_0^1 = \frac{1}{2}\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{1}{2}\left(\frac{15-10+3}{15}\right) = \boxed{\frac{4}{15}}$

$\int_0^1\int_0^{1-x}\underbrace{\int_0^{x^{1/2}} dy}_{y\big|_{y=0}^{y=x^{1/2}} = x^{1/2}}\,dz\,dx = \int_0^1\underbrace{\int_0^{1-x} x^{1/2}\,dz}_{x^{1/2}z\big|_{z=0}^{z=1-x} = x^{1/2}(1-x)}\,dx$

$= \int_0^1 x^{1/2}(1-x)\,dx = \cdots = \boxed{\frac{4}{15}}$ (see first integral above)

③ a) $\underbrace{\overbrace{x^2+y^2}^{r^2}+z^2}_{\rho^2} = 3 \to \boxed{\begin{matrix} r^2+z^2 = 3 \\ \rho = \sqrt{3}\end{matrix}} \xrightarrow[z]{\text{solve for}}$

$\underbrace{x^2+y^2}_{r^2}+(z-1)^2 = 1$

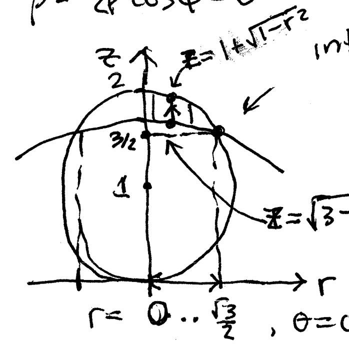
$\downarrow$

$\underbrace{x^2+y^2+z^2}_{\rho^2} - \underbrace{2z}_{\rho\cos\phi} + \cancel{1} = \cancel{1}$

$\rho^2 - 2\rho\cos\phi = 0$

$\boxed{\begin{matrix} r^2+(z-1)^2 = 1 \\ \\ \rho = 2\cos\phi \end{matrix}} \xrightarrow[\text{for } z]{\text{solve}}$ (over)

b)



$r = 0..\frac{\sqrt{3}}{2}$, $\theta = 0..2\pi$

intersection: $r^2+z^2 = 3$

$\underline{r^2+z^2-2z+1 = 1}$

$3 - 2z + \cancel{1} = \cancel{1} \to \boxed{z = 3/2}$ smallest value of z:

$r = \sqrt{3-z^2} = \sqrt{3-\frac{9}{4}} = \frac{\sqrt{3}}{2}$

$\boxed{r = \frac{\sqrt{3}}{2}}$ largest value of r.

MAT2500-01/04 12S TEST 3 Takehome: Answers (2)

(3) a) continued $r^2+z^2=3 \to z=\pm\sqrt{3-r^2}$, $z\ge 0$: $z=\sqrt{3-r^2}$

$r^2+(z-1)^2=1 \to (z-1)=\pm\sqrt{1-r^2}$

$z=1\pm\sqrt{1-r^2}$ → upper root: $\boxed{z=1+\sqrt{1-r^2}}$

c) $$V=\int_0^{2\pi}\int_0^{\frac{\sqrt3}{2}}\underbrace{\int_{\sqrt{3-r^2}}^{1+\sqrt{1-r^2}} 1\cdot r\,dz}\,dr\,d\theta$$

radial integral: $rz\Big|_{z=\sqrt{3-r^2}}^{z=1+\sqrt{1-r^2}} = [1+(1-r^2)^{1/2}-(3-r^2)^{1/2}]\,r$

$$-\frac12\int_0^{\frac{\sqrt3}{2}}[1+(1-r^2)^{1/2}-(3-r^2)^{1/2}](-2)r\,dr$$ $-du$ if $u=c-r^2$

$$\frac{r^2}{2}-\frac12\frac{u^{3/2}}{3/2}+\frac12\frac{u^{3/2}}{3/2}$$

$$=\frac{r^2}{2}-\frac13(1-r^2)^{3/2}+\frac13(3-r^2)^{3/2}\Big|_0^{\sqrt3/2}$$

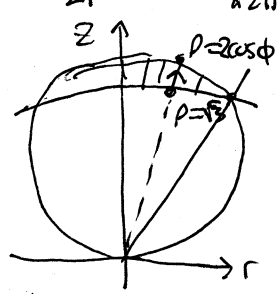
$$=\frac38-\frac13\underbrace{\left(\frac14\right)^{3/2}}_{1/8}+\frac13\underbrace{\left(\frac94\right)^{3/2}}_{\frac{27}{8}}+\frac13-\frac13\underbrace{3^{3/2}}_{3^{1/2}}$$

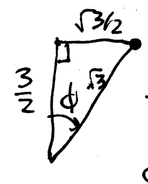
$$=\frac{1}{3\cdot8}(9-1+27+8)-\sqrt3$$

$$=\frac{43}{24}-\sqrt3 \xrightarrow{\theta\text{ integral}} \times 2\pi$$

$$\boxed{V=\left(\frac{43}{24}-\sqrt3\right)2\pi \approx 0.3746}$$

d)





intersection pt: $\tan\phi=\frac{\sqrt3/2}{3/2}=\frac{1}{\sqrt3}$

$\phi=\pi/6$

$\phi=0..\frac{\pi}{6}$, $\theta=0..2\pi$

e) $$V=\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\underbrace{\int_{\sqrt3}^{2\cos\phi}1\,\rho^2\sin\phi\,d\rho}\,d\phi\,d\theta$$

$$\frac{\rho^3}{3}\sin\phi\Big|_{\rho=\sqrt3}^{\rho=\cos\phi}=\frac13(\cos^3\phi-3\sqrt3)\sin\phi$$

θ integral

$$=(2\pi)\int_0^{\frac{\pi}{6}}\frac13(8\underbrace{\cos^3\phi}_{u^3}-3\sqrt3)\underbrace{\sin\phi\,d\phi}_{-du}$$

$$=\frac{2\pi}{3}\left(-\frac{8\cos^4\phi}{4}+\cos\phi\,3\sqrt3\right)\Big|_0^{\pi/6}$$

$$=\frac{2\pi}{3}\left(-\left(\frac{3^{1/2}}{2}\right)^4\frac84+3\,3^{1/2}\frac{3^{1/2}}{2}+\frac84-3\sqrt3\right)=2\pi\left(\frac{43}{24}-\sqrt3\right)\ \checkmark$$

$$-\frac{9}{64}+\cancel{\frac{1\cdot16}{4\cdot16}}+\frac{9\cdot32}{2\cdot32}\ 3\sqrt3 \longrightarrow \text{maple}$$

f) agrees with Maple

all agree with maple!!

$$\boxed{\frac{4}{15}\sqrt2-\frac13 \approx 0.04379}$$

(3g) $$V_z=\int_0^{2\pi}\int_0^{\frac{\sqrt3}{2}}\underbrace{\int_{\sqrt{3-r^2}}^{1+\sqrt{1-r^2}} z\,r\,dz}\,dr\,d\theta$$

$$=2\pi\int_0^{\sqrt3/2} r\frac{z^2}{2}\Big|_{z=\sqrt{3-r^2}}^{z=1+\sqrt{1-r^2}}dr$$

$$=(2\pi)\int_0^{\sqrt3/2}\frac r2\left[1+2(1-r^2)^{1/2}+(1-\cancel{r^2})-(3-\cancel{r^2})\right]dr$$

$$=2\pi\int_0^{\sqrt3/2}\left[-\frac r2+(1-r^2)^{1/2}r\right]dr$$

$$=2\pi\left[-\frac{r^2}{4}+\left(-\frac12\right)\frac{(1-r^2)^{3/2}}{3/2}\right]\Big|_0^{\sqrt3/2}$$

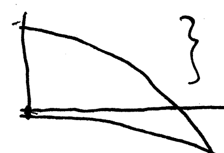
$$=2\pi\left[-\frac{3}{16}-\frac13\left(\frac14\right)^{3/2}+\frac13\right]$$

$$=2\pi\left[\frac{7}{48}-\frac{1}{24}\right]=\boxed{\frac{5}{24}\pi \approx 0.6545}$$

$$\bar z=\frac{\frac{5\pi}{24}}{2\pi\left(\frac{43}{24}-\sqrt3\right)}=\boxed{\frac{5}{2(43-24\sqrt3)}\approx 1.747}$$

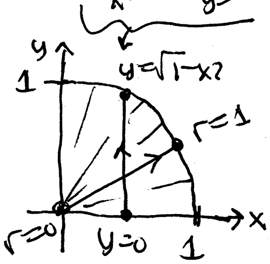
≈ 1.75 with Maple 5 digit evaluation (surprising but true)
$\neq 7/4$ (rational & irrational number cannot be equal)

compare to $\sqrt3\approx 1.732$ just a hair above the north pole of the inner sphere! I would have expected it to be higher.



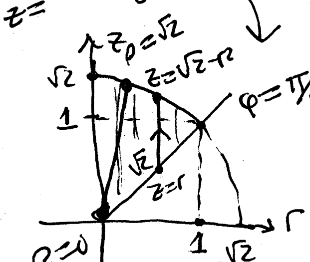
but the moments must balance.

(4) a) $$\int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}}\int_{z=\sqrt{x^2+y^2}=r}^{z=\sqrt{2-x^2-y^2}=\sqrt{2-r^2}} xy\,dz\,dy\,dx$$



$\theta=0..\pi/2$

$r=0..1$



$\rho=0..\sqrt2$

$\phi=0..\pi/4$

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt2}\Big[\underbrace{\overbrace{(\rho\sin\phi)}^{r}\cos\theta}_{x}\Big]\Big[\underbrace{\overbrace{(\rho\sin\phi)}^{r}\sin\theta}_{y}\Big]\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt2}\rho^4\sin^3\phi\,\sin\theta\cos\theta\,d\rho\,d\phi\,d\theta$$

$$=\underbrace{\int_0^{\sqrt2}\rho^4d\rho}\ \underbrace{\int_0^{\pi/4}\sin^3\phi\,d\phi}\ \underbrace{\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta}$$

$$\frac{\rho^5}{5}\Big|_0^{\sqrt2}=\frac{4\sqrt2}{5}\qquad \frac{1}{12}(8-5\sqrt2)\qquad \frac12$$