MAT2500-01/04 TEST 2 Answers (1)

① $f(x,y) = \frac{5x}{x^2+y^2} = 5x(x^2+y^2)^{-1}$, $f(1,2) = \frac{5}{1+4} = 1$

a) $f_x = \frac{5[(x^2+y^2)(1) - x(2x)]}{(x^2+y^2)^2} = \frac{5(y^2-x^2)}{(x^2+y^2)^2}$

$f_y = 5x(-1)(x^2+y^2)^{-2}(2y) = \frac{-10xy}{(x^2+y^2)^2}$

$\vec\nabla f(x,y) = \frac{5\langle y^2-x^2, -2xy\rangle}{(x^2+y^2)^2}$

$\vec\nabla f(1,2) = \frac{5\langle 4-1, -4\rangle}{5^2} = \frac{\langle 3,-4\rangle}{5}$

$|\vec\nabla f(1,2)| = \frac15\sqrt{3^2+4^2} = \frac55 = 1$ unitvector.

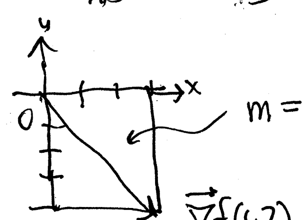
$\boxed{\hat u = \langle \frac35, \frac{-4}{5}\rangle = \frac{\langle 3,-4\rangle}{5}}$ gives direction of most rapid increase.

b) $\vec v = \langle 2,-1\rangle$, $\hat v = \frac{\langle 2,-1\rangle}{\sqrt5}$

$D_{\hat v} f(1,2) = \hat v \cdot \vec\nabla f(1,2)$

$= \frac{\langle 2,-1\rangle}{\sqrt5}\cdot\frac{\langle 3,-4\rangle}{5} = \frac{10}{5\sqrt5} = \boxed{\frac{2}{\sqrt5}}$

c)




$m = \boxed{-\frac43 = \text{slope}}$

$\vec\nabla f(1,2)$

normal line is along gradient.

$f(1,2) = \frac{5(1)}{1+4} = 1$

$\boxed{\frac{5x}{x^2+y^2} = 1 \quad \text{or} \quad x^2+y^2 = 5x}$

circle with center on x-axis.

d) $\boxed{L(x,y)} = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$\boxed{= 1 + \frac35(x-1) - \frac45(y-2)}$

e) $f(.98, 2.01) \approx L(.98, 2.01) = 1 + \frac35(.98-1) - \frac45(2.01-2)$

$= 1 + \frac35(-.02) - \frac45(.01) = 1 \; \frac{-.04-.06}{5} =$

$= 1 - .02 = \boxed{0.98}$

OPTIONAL:

④ c): $\frac{(-2-t)^2}{4} + (1+2t)^2 + \frac{(-3-2t)^2}{3} = 5$

↳ Maple: $t = 0, \boxed{-\frac{108}{67}}$ ↳ subs in $\vec r(t)$: $\vec r(-\frac{108}{67}) = \frac{1}{67}\langle -26, -149, 15\rangle$ ✓

② $f(x,y) = x^3 - 3x - y^3 + 12y$

a) $f_x = 3x^2 - 3 = 3(x^2-1) = 0 \to x^2 = \pm 1$

$f_y = -3y^2 + 12 = 3(-y^2 + \ ) = 0 \to y^2 = \pm 2$

$f_{xx} = 6x$

$f_{yy} = -6y$

$f_{xy} = 0$

$f_x(1,2) = 3(1-1) = 0$ ✓ $\quad f_x(-1,2) = 3(1-1) = 0$ ✓

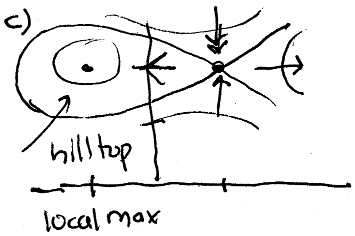
$f_y(1,2) = 3(4-4) = 0$ ✓ $\quad f_y(-1,2) = 3(4-4) = 0$ ✓

(1,2), (−1,2) are critical pts.

b)

| | (1,2) | (−1,2) | |
|---|---|---|---|
| $f_{xx}$ | 6 | −6 | local max? |
| $f_{yy}$ | −12 | −12 | |
| $f_{xy}$ | 0 saddle | 0 | confirms local max. |
| $f_{xx}f_{yy} - f_{xy}^2$ | $-72 < 0$ saddle | $72 > 0$ | |

c)




increases in arrow directions so saddle.

③ $\frac{\partial}{\partial z}(yz + x\ln y = z^2 + 1)$ $\quad y = y(x,z)$ dep var.

$\frac{\partial y}{\partial z} z + y(1) + x\frac1y\frac{\partial y}{\partial z} = 2z$, $\quad \frac{\partial y}{\partial z}(\frac{x}{y} + z) = 2z - y$

$\frac{\partial y}{\partial z} = \frac{2z-y}{\frac{x}{y}+z} = \frac{y(2z-y)}{x+yz}$ $\quad \frac{\partial y}{\partial z}\Big|_{(1,e,e)} = \frac{e(2e-e)}{1+e^2} = \boxed{\frac{e^2}{1+e^2}}$

④ $F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{3} = 5$

a) $\vec\nabla F = \langle \frac{x}{2}, 2y, \frac23 z\rangle$, $\vec\nabla F(2,1,-3) = \boxed{\langle -1, 2, -2\rangle = \vec n}$

$0 = \vec n\cdot(\vec r - \vec r_0) = \langle -1,2,-2\rangle\cdot\langle x+2, y-1, z+3\rangle$

$= -(x+2) + 2(y-1) - 2(z+3)$

b) $\vec r = \vec r_0 + t\vec n$

$\langle x,y,z\rangle = \langle -2,1,-3\rangle + t\langle -1,2,-2\rangle$

$= \langle -2-t, 1+2t, -3-2t\rangle$ normal line

$= -x - 2 + 2y - 2 - 2z - 6$

$= \boxed{-x + 2y - 2z - 10 = 0}$

$\boxed{\text{or} \quad x - 2y + 2z = -10}$

# MAT2500-01/04 TEST2 Answers Optional Question (2)

(5) contour = level curve thru $(x_0,y_0)$:

$$\frac{\cancel{5}x}{x^2+y^2} = \frac{\cancel{5}x_0}{x_0^2+y_0^2} \to x(x_0^2+y_0^2) = x_0(x^2+y^2)$$

$$x^2+y^2 = \frac{x}{x_0}(x_0^2+y_0^2)$$

$$x^2 - \frac{x}{x_0}(x_0^2+y_0^2) + y^2 = 0$$

$$\left(x - \frac{1}{2x_0}(x_0^2+y_0^2)\right)^2 - \left(\frac{1}{x_0}(x_0^2+y_0^2)\right)^2 + y^2 = 0$$

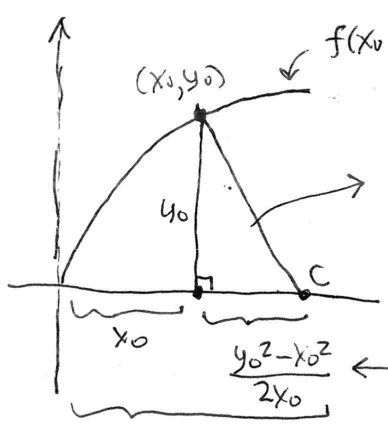
$$(x-x_c)^2 + y^2 = r_0^2$$

$$x_c = \frac{1}{2x_0}(x_0^2+y_0^2), \quad r_0 = \frac{(x_0^2+y_0^2)}{x_0}$$

center located at $(0, x_c)$ on x-axis.
radius $r_0$.

If you look at the contour plot you see that all these circles go through the origin where the function has to be discontinuous.

$f(x_0,y_0) = f(x,y)$




$$\therefore \text{ slope } m = \frac{-y_0}{\left(\frac{y_0^2-x_0^2}{2x_0}\right)} = \frac{-2x_0y_0}{y_0^2-x_0^2}$$

Note $x_c = \frac{1}{2}x_0 + \frac{y_0^2}{2x_0} = x_0 + \frac{y_0^2}{2x_0} - \frac{x_0}{2}$

$= x_0 + \frac{y_0^2-x_0^2}{2x_0}$

$= \frac{x_0^2+y_0^2}{2x_0}$

## 4c continued by hand

$$\frac{3(2+t)^2 + 12(1+2t)^2 + 4(3+2t)^2}{3\cdot 4} = 5$$

numer $= 3(4+4t+t^2)$
$+ 12(1+4t+4t^2)$
$+ 4(9+12t+4t^2)$

$= (12+12+36) + (12+48+48)t + (3+48+16)t^2$

$= 60 + 108t + 67t^2 \underset{\text{RHS}}{=} 3\cdot 4\cdot 5 = 60$

$67t^2 + 108t = (67t+108)t = 0$

$t = 0, -\frac{108}{67}$

$(x,y,z) = \langle -2+\frac{108}{67}, 1+2(-\frac{108}{67}), -3-(-\frac{108}{67})\rangle$

$= \frac{\langle 108-2(67), 67-2(108), 108-3(67)\rangle}{67}$

$= \frac{\langle -26, -149, 15\rangle}{67}$ ✓ you can check that indeed this satisfies the ellipsoid eqn

moral: if you know what steps to take, Maple can easily implement them when handwork becomes torture.

compare with gradient:

$\vec{\nabla} f(x_0,y_0) = 5\langle y_0^2-x_0^2, -2x_0y_0\rangle$

ratio should give slope:

$m = \frac{-2x_0y_0}{y_0^2-x_0^2}$ ✓ checks!