MAT2500-01/04 12S Test 1 Answers

a) $\vec{r} = \langle e^t\cos t, e^t\sin t, e^t \rangle$

$\boxed{\vec{r}\,' = \langle e^t\cos t - e^t\sin t,\ e^t\sin t + e^t\cos t,\ e^t \rangle}$

$\vec{r}\,'' = \langle e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t,$
$\quad e^t\sin t + e^t\cos t + e^t\cos t - e^t\sin t,\ e^t \rangle$

$= \boxed{\langle -2e^t\sin t,\ 2e^t\cos t,\ e^t \rangle}$

$|\vec{r}\,'|^2 = (e^t)^2\left((\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1\right)$

$= (e^t)^2\left(\begin{matrix}\cos^2 t + \sin^2 t - 2\cos t\sin t \\ + \cos^2 t + \sin^2 t + 2\cos t\sin t\end{matrix} + 1\right)$

$= 3(e^t)^2 \qquad \boxed{|\vec{r}\,'| = \sqrt{3}\,e^t}$

$\hat{T}(t) = \dfrac{\vec{r}\,'}{|\vec{r}\,'|} = \dfrac{e^t\langle \cos t - \sin t, \cos t + \sin t, 1\rangle}{\sqrt{3}e^t}$

$= \boxed{\frac{1}{\sqrt{3}}\langle \cos t - \sin t, \cos t + \sin t, 1\rangle}$

$|\vec{r}\,''| = e^t\sqrt{4\sin^2 t + 4\cos^2 t + 1} = \boxed{\sqrt{5}\,e^t}$

$\vec{r}(0) = \langle 1, 0, 1\rangle$

$\vec{r}\,'(0) = \langle 1, 1, 1\rangle \qquad |\vec{r}\,'(0)| = \sqrt{3}$

$\vec{r}\,''(0) = \langle 0, 2, 1\rangle \qquad |\vec{r}\,''(0)| = \sqrt{5}$

$\hat{T}(0) = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle$

d) $\vec{r} = \vec{r}(0) + t\,\vec{r}\,'(0)$

$\boxed{\langle x, y, z\rangle = \langle 1, 0, 1\rangle + t\langle 1, 1, 1\rangle = \langle 1+t, t, 1+t\rangle}$

$\boxed{\text{or } x = 1+t,\ y = t,\ z = 1+t}$

e) $\kappa = \dfrac{\sqrt{6}e^{2t}}{(\sqrt{3}e^t)^3} = \dfrac{\sqrt{6}\,e^{2t}}{3\sqrt{3}\,e^{3t}} = \boxed{\frac{\sqrt{2}}{3}e^{-t}}$

$\boxed{\rho = \frac{3}{\sqrt{2}}e^t} \qquad \left(= \frac{1}{\kappa}\right)$

f) $T' = \frac{1}{\sqrt{3}}\langle -\sin t - \cos t, -\sin t + \cos t, 0\rangle$

$|T'| = \frac{1}{\sqrt{3}}\sqrt{(s+c)^2 + (c-s)^2}$
$\quad s^2 + c^2 + 2sc + c^2 + s^2 - 2sc$

$= \frac{\sqrt{2}}{\sqrt{3}}$

$\hat{N} = \dfrac{\hat{T}'}{|\hat{T}'|} = \boxed{\frac{1}{\sqrt{2}}\langle -\sin t - \cos t, -\sin t + \cos t, 0\rangle}$

g) $L = \int_{-1/2}^{1/2} |\vec{r}\,'(t)|\,dt = \int_{-1/2}^{1/2} \sqrt{3}e^t\,dt = \sqrt{3}e^t \Big|_{-1/2}^{1/2}$

$= \boxed{\sqrt{3}(e^{1/2} - e^{-1/2}) \approx 1.805}$

b) $\vec{r}\,'\times\vec{r}\,'' = e^t\langle c-s, c+s, 1\rangle \times e^t\langle -2s, 2c, 1\rangle$

$= e^{2t}\begin{vmatrix} i & j & k \\ c-s & c+s & 1 \\ -2s & 2c & 1\end{vmatrix} = e^{2t}\langle \underbrace{c+s-2c}_{s-c}, \underbrace{-2s-(c-s)}_{-s-c}, \underbrace{(c-s)(2c)+2s(c+s)}_{2c^2 - 2sc + 2sc + 2s^2 = 2}\rangle$

$= e^{2t}\langle \sin t - \cos t, -\sin t - \cos t, 2\rangle$

$|\vec{r}\,'\times\vec{r}\,''| = e^{2t}\sqrt{(s-c)^2 + (s+c)^2 + 4} = \sqrt{6}\,e^{2t}$
$\quad \underbrace{s^2 + c^2}_{1} - 2sc + \underbrace{s^2 + c^2}_{1} + 2sc$

$\hat{B} = \dfrac{\vec{r}\,'\times\vec{r}\,''}{|\vec{r}\,'\times\vec{r}\,''|} = \boxed{\frac{1}{\sqrt{6}}\langle \sin t - \cos t, -\sin t - \cos t, 2\rangle}$

$\vec{B}(0) = \frac{1}{\sqrt{6}}\underbrace{\langle -1, -1, 2\rangle}_{\vec{n}} \qquad r_0 = \vec{r}(0) = \langle 1, 0, 1\rangle$

h) $a_T(0) = \hat{T}(0)\cdot\vec{r}\,''(0)$

$= \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle\cdot\langle 0, 2, 1\rangle = \frac{2+1}{\sqrt{3}} = \boxed{\sqrt{3}}$

$a_N(0) = \hat{N}(0)\cdot\vec{r}\,''(0)$

$= \frac{1}{\sqrt{2}}\langle -1, 1, 0\rangle\cdot\langle 0, 2, 1\rangle = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}$

$a_T(0)^2 + a_N(0)^2 = 3 + 2 = 5$

$a(0)^2 = 5 \checkmark$

c) $\vec{n}\cdot(\vec{r} - \vec{r}_0) = 0: \quad \langle -1, -1, 2\rangle\cdot\langle x-1, y-0, z-1\rangle = 0$

$-(x-1) - (y) + 2(z-1) = 0$

$-x + 1 - y + 2z - 2 = 0 \qquad \boxed{-x - y + 2z - 1 = 0}$

$\quad$ or $x + y - 2z = -1$

i) $\vec{C}(0) = \vec{r}\,''(0) + \rho(0)\,\hat{N}(0)$

$= \langle 1, 0, 1\rangle + \frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle -1, 1, 0\rangle$

$= \langle 1, 0, 1\rangle + \frac{3}{2}\langle -1, 1, 0\rangle$

$= \langle -\frac{1}{2}, \frac{3}{2}, 1\rangle \checkmark$

j) $a_T(t) = \hat{T}(t)\cdot\vec{r}\,''(t) = \frac{1}{\sqrt{3}}\langle c-s, c+s, 1\rangle\cdot e^t\langle -2s, 2c, 1\rangle$

$= \frac{e^t}{\sqrt{3}}(-2sc + 2s^2 + 2c^2 + 2cs + 1) = \sqrt{3}e^t = |\vec{r}\,'|$ so $\boxed{a_T(t) = |\vec{r}\,'(t)|}$ (obvious from $a_T = v' = (\sqrt{3}e^t)' = \sqrt{3}e^t = v$)