MAT2705-01/02 12F Final Exam Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC), REALLY. You are encouraged to use technology to check all of your hand results.

A certain physical system has the following equations of motion and initial conditions:

$$\vec{x}''(t) = A\,\vec{x}(t) + \vec{F}(t),\ \vec{x}(0) = \vec{0},\ \vec{x}'(0) = \vec{0} \text{ where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} -10 & 2 \\ 3 & -15 \end{bmatrix}, \vec{F}(t) = \begin{bmatrix} -42\cos(2t) \\ 21\cos(2t) \end{bmatrix}$$

a) Rewrite this system of DEs ***and*** its initial conditions as 6 scalar equations. Solve this system with Maple and write down the solutions for the two unknowns.

b) By hand showing all steps, find the smallest integer component eigenvectors $\vec{b}_1, \vec{b}_2$ of the coefficient matrix A produced by the solution algorithm after rescaling of the standard results by positive multiples if necessary, ordered so that the corresponding eigenvalues satisfy $|\lambda_1| < |\lambda_2|$. Evaluate the matrix $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$ and its inverse, and the diagonalized matrix $A_B = B^{-1} A\, B$. [Use technology to check that your inverse is correct.]

c) What are the slopes m_1, m_2 of the lines through the origin containing the two eigenvectors? On the grid provided, draw in those two lines, labeling them by their corresponding coordinates y_1, y_2 in the positive direction determined by the eigenvectors and then indicate by thicker arrows both eigenvectors $\vec{b}_1, \vec{b}_2$, labeled by their symbols. Recall $\vec{x} = B\,\vec{y}$, $\vec{y} = B^{-1}\,\vec{x}$, where $\vec{y} = \langle y_1, y_2 \rangle$. Also label the x_1, x_2 axes.

d) Find by hand the general solution of the corresponding decoupled system of DEs $\vec{y}\,'' = A_B \vec{y} + B^{-1}\vec{F}$. First write them out in matrix form, then obtain the two equivalent scalar DEs which are its components. Then solve them to find their general solutions using the method of undetermined coefficients, identifying the homogeneous and particular parts of each solution: $y_1 = y_{1h} + y_{1p}$, $y_2 = y_{2h} + y_{2p}$. State your solution in scalar form and box it: $y_1(t) = \ldots, y_2(t) = \ldots$.

e) Then express the general solution for $\vec{x} = B\,\vec{y}$ ***and impose the initial conditions***. Write out and box the scalar solutions: $x_1(t) = \ldots, x_2(t) = \ldots$. Do they agree with Maple's solution? If not, look for your error. Did you input the equations correctly?

f) Express your (correct) solution as a sum of the two eigenvector modes and the response mode in the form: $\vec{x} = y_{1h}\,\vec{b}_1 + y_{2h}\,\vec{b}_2 + \cos(2t)\,\vec{b}_3$ thus identifying the particular solution $\vec{x}_p$ (last term), the response vector coefficient $\vec{b}_3$ and the homogeneous solution $\vec{x}_h$ (first two terms). What are the values of the natural frequencies ω_1 and ω_2 associated with the homogeneous solution? Which frequency is associated with the tandem mode (same signed eigenvector components) and which with the accordian mode (opposite signed eigenvector components)?

g) On the grid provided, draw in the vector $\vec{x}_h(0)$ and $\vec{x}_p(0)$ and the new axes of the coordinates $\{y_1, y_2\}$ and label the vectors and all 4 axes properly. What are the new coordinates $\langle y_{1h}(0), y_{2h}(0) \rangle$? On your graph, draw in the parallelogram parallel to the new coordinate axes which projects the vector $\vec{x}_h(0)$ along those axes and identify the sides of the parallelogram on those axes by the number of multiples of the corresponding eigenvector, i.e., $y_{1h}(0)\,\vec{b}_1$ and $y_{2h}(0)\,\vec{b}_2$. Do these seem consistent with your plot? Draw in the double arrow through the origin corresponding to the oscillation in each of the three separate modes from endpoint to endpoint.

h) Suppose we remove the driving function and release the springs with rotational motion about the origin:
$x_1''(t) = -10\,x_1(t) + 2\,x_2(t),\ x_2''(t) = 3\,x_1(t) - 15\,x_2(t),\ x_1(0) = 1,\ x_2(0) = 1,\ x_1'(0) = -1,\ x_2'(0) = 1$.
Draw a simple labeled diagram illustrating the initial data with $\vec{x}(0)$, and $\vec{x}'(0)$ plotted with its tail at the tip of $\vec{x}(0)$. Then use Maple to find the solution of this IVP and examine the faster frequency mode contributions to the solutions for x_1 and x_2 (simply ignore the slower mode terms in the solution). Write each of these sinusoidal functions in phase-shifted cosine form stating explicitly (A_1, δ_1) and (A_2, δ_2) respectively, making a single completely labeled diagram that supports your work, showing both coefficient vectors on the same axes. Evaluate the the quotient A_2/A_1 and difference $\delta_2 - \delta_1$? Explain. What is the period $T_{accordian}$ for this faster mode?

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have not accessed any of the class web pages or any other sites during the exam. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: