MAT2705-01/02 12F TEST 3 Answers (1)

① a) $25x'' + 10x' + 26x = F$

$x'' + \underbrace{\tfrac{2}{5}}_{k_0} x' + \underbrace{\tfrac{26}{25}}_{\omega_0^2} x = F/25$

$\omega_0 = \boxed{\frac{\sqrt{26}}{5}} \approx 1.0198 \approx \boxed{1.020}$

$k_0 = \boxed{2/5 = 0.4}$

$\tau_0 = 1/k_0 = \boxed{5/2 = 2.5}$

$Q = \omega_0\tau_0 = \boxed{\frac{\sqrt{26}}{5}}$ ≈ 2.54951 $\approx \boxed{2.550} > \frac{1}{2}$ underdamped

$T_0 = \frac{2\pi}{\sqrt{26}/5} = \boxed{\frac{10\pi}{\sqrt{25}} \approx 6.16}$

b) $25x'' + 10x' + 26x = 0$

$x = e^{rt} \to 25r^2 + 10r + 26 = 0$

$r = -\frac{1}{5} \pm i$, $x = e^{-\frac{1}{5}t} e^{\pm it} = e^{-1/5t}(\cos t \pm i \sin t)$

real soln basis: $\{e^{-1/5t}\cos t,\ e^{-\frac{1}{5}t}\sin t\}$

gen hom soln: $\boxed{x_h = e^{-\frac{t}{5}}(c_1\cos t + c_2 \sin t)}$

$\boxed{\omega_1 = 1}$ unit frequency $\qquad \tau_1 = 5$

$T_1 = 2\pi/\omega_1 = \boxed{2\pi}$

Note $5\tau_1/T_1 = 3.98$ ~ about # oscillations in decay window

c) $25x'' + 10x' + 26x = \underline{10\sin t}$

soln of $(D^2+1)x = 0$

$\therefore$ trial function

$26\,[\,x_p = c_3\cos t + c_4 \sin t$

$10\,[\,x_p' = -c_3\sin t + c_4\cos t$

$25\,[\,x_p'' = -c_3\cos t - c_4\sin t$

$25x_p'' + 10x_p' + 26x_p = [(26-25)c_3 + 10c_4]\cos t + [-10c_3 + (26-25)c_4]\sin t = 10\sin t$

$\begin{bmatrix}1 & 10\\ -10 & 1\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}0\\ 10\end{bmatrix} \to \begin{bmatrix}c_3\\ c_4\end{bmatrix} = \frac{1}{101}\begin{bmatrix}1 & -10\\ 10 & 1\end{bmatrix}\begin{bmatrix}0\\ 10\end{bmatrix} = \frac{10}{101}\begin{bmatrix}-10\\ 1\end{bmatrix}$

$\boxed{x_p = -\frac{100}{101}\cos t + \frac{10}{101}\sin t}$

$x = x_h + x_p = e^{-t/5}(c_1\cos t + c_2\sin t) + \frac{1}{101}(-100\cos t + 10\sin t)$

$x' = -\frac{1}{5}e^{-t/5}(c_1\cos t + c_2\sin t) + e^{-t/5}(-c_1\sin t + c_2\cos t) + \frac{1}{101}(+100\sin t + 10\cos t)$

$x(0) = c_1 - \frac{100}{101} = 0 \qquad c_1 = \frac{100}{101}$

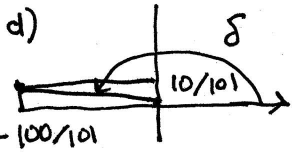
$x'(0) = -\frac{1}{5}c_1 + c_2 + \frac{10}{101} = 0 \qquad c_2 = \frac{1}{5}\left(\frac{100}{101}\right) - \frac{10}{101} = \frac{10}{101}$

$\boxed{x = \underbrace{\frac{1}{101}(-100\cos t + 10\sin t)}_{x_{ss}} + \underbrace{\frac{1}{101}e^{-t/5}(100\cos t + 10\sin t)}_{}}$

$x_{transient}$: $A_{trans} = \frac{10}{101}\sqrt{10^2+1^2} = \frac{10}{\sqrt{101}} \approx 0.9950$

e) envelope: $\boxed{x = \pm\frac{10}{\sqrt{101}}e^{-t/5}}$

d)



$A = \frac{1}{101}\sqrt{(10)^2 + (-100)^2} = \frac{10}{101}\sqrt{1^2 + 100} = \boxed{\frac{10}{\sqrt{101}} \approx 0.995}$

$\delta = \pi - \arctan(1/10) \approx 3.042 \approx 174.3° \approx 0.484$ cycles

$\delta - \pi/2 = \boxed{\pi/2 - \arctan(1/10) \approx 1.471 \approx 84.3° \approx 0.234 \text{ cycles}}$

e) positive phase shift difference → x_{ss} shifted right on graph nearly 1/4 cycle, later in time (lags)

f) $25x'' + 10x' + 26x = \underline{F_0\sin\omega t}$

$26\,[\,x_p = c_3\cos\omega t + c_4\sin\omega t$

$10\,[\,x_p' = -\omega c_3\sin\omega t + \omega c_4\cos\omega t$

$25\,[\,x_p'' = -\omega^2 c_3\cos\omega t - \omega^2 c_4\sin\omega t$

$25x_p'' + 10x_p' + 26x_p = [(26-25\omega^2)c_3 + 10\omega c_4]\cos\omega t + [-10\omega c_3 + (26-25\omega^2)c_4]\sin\omega t = F_0\sin\omega t.$

$\begin{bmatrix}26-25\omega^2 & 10\omega\\ -10\omega & 26-25\omega^2\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}0\\ F_0\end{bmatrix}$

$\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \frac{\begin{bmatrix}26-25\omega^2 & -10\omega\\ 10\omega & 26-25\omega^2\end{bmatrix}\begin{bmatrix}0\\ F_0\end{bmatrix}}{(26-25\omega^2)^2 + (10\omega)^2} = \frac{F_0\begin{bmatrix}-10\omega\\ 26-25\omega^2\end{bmatrix}}{(26-25\omega^2)^2 + 100\omega^2}$

$\boxed{x_p = \frac{F_0\left[(-10\omega)\cos\omega t + (26-25\omega^2)\sin\omega t\right]}{(26-25\omega^2)^2 + 100\omega^2}}$

g) $\boxed{A(\omega) = F_0\sqrt{(-10\omega)^2 + (26-25\omega^2)^2}\,/\,(26-25\omega^2)^2 + 100\omega^2 = \frac{F_0}{\sqrt{(26-25\omega^2)^2 + 100\omega^2}}} \equiv \frac{F_0}{\mathcal{D}^{1/2}}$

$0 = A'(\omega) = -\frac{F_0}{2}\mathcal{D}^{-1/2}\mathcal{D}'(\omega)$

$= -\frac{F_0}{2}\mathcal{D}^{-1/2}\underbrace{\left[2(26-25\omega^2)(-50\omega) + 200\omega\right]}_{100\omega\,[25\omega^2 - 26 + 2]}$

$25\omega^2 - 24 = 0$

$\omega^2 = \frac{24}{25}$ $\boxed{\omega_p = \frac{2}{5}\sqrt{6} \approx 0.9797 \approx 0.980}$

$A(\omega_p) = \frac{F_0}{\sqrt{(26-24)^2 + 96}} = \boxed{\frac{F_0}{10}}$

$\frac{A(\omega_p)}{A(0)} = \frac{F_0/10}{F_0/\sqrt{26}} = \frac{\sqrt{26}}{10} \approx 2.6 \leftrightarrow Q \approx 2.55$

$F_0 = 10$: $A(1) = \frac{10}{\sqrt{1^2+100}} = \frac{10}{\sqrt{101}}$ ✓

→ very close!

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① h) $\omega = 0..10$ seems to give a nice view showing the asymptote at $\omega >> 1$ and the nice peak near $\omega = 1$.

② a) $(k_1,k_2,k_3) = \left(\frac{5}{10},\frac{5}{25},\frac{5}{10}\right) = \left(\frac12,\frac15,\frac12\right)$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}' = \underbrace{\begin{bmatrix}-\frac12 & 0 & \frac12\\ \frac12 & -\frac15 & 0\\ 0 & \frac15 & -\frac12\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \qquad \begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{bmatrix} = \begin{bmatrix}0\\9\\0\end{bmatrix}$$

b)

$$0 = |A-\lambda I| = \begin{vmatrix}-\frac12-\lambda & 0 & \frac12\\ \frac12 & -\frac15-\lambda & 0\\ 0 & \frac15 & -\frac12-\lambda\end{vmatrix} \overset{\text{Maple}}{=} -\lambda^3 - \frac65\lambda^2 - \frac{9}{20}\lambda$$

$$= -\lambda\underbrace{\left(\lambda^2+\frac65\lambda+\frac{9}{20}\right)}_{\text{quad formula}} = \cdots \xrightarrow[\text{maple}]{} \lambda = -\frac35 \pm \frac{3}{10}i,\ 0$$

$$\lambda = 0:\ A = \begin{bmatrix}-\frac12 & 0 & \frac12\\ \frac12 & -\frac15 & 0\\ 0 & \frac15 & -\frac12\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1\\ 1 & -2/5 & 0\\ 0 & 2/5 & -1\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1\\ 1 & 0 & -1\\ 0 & 1 & -5/2\end{bmatrix}$$

$$\overset{\text{LLF}}{\to} \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -5/2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \begin{aligned}x_1 &= t\\ x_2 &= 5/2\,t\\ x_3 &= t\end{aligned} \qquad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = t\underbrace{\begin{bmatrix}1\\5/2\\1\end{bmatrix}}_{\vec b_1}$$

$$\lambda = -\frac35 + \frac{3i}{10}:\ A - \lambda I = \begin{bmatrix}\overbrace{-\frac12+\frac35}^{\frac{1}{10}}-\frac{3i}{10} & 0 & \frac12\\ \frac12 & -\frac15+\frac35-\frac{3i}{10} & 0\\ 0 & \frac15 & -\frac12+\frac35-\frac{3i}{10}\end{bmatrix} \to$$

$\frac25$

$$\begin{bmatrix}1-3i & 0 & 5\\ 1 & \frac45-\frac35 i & 0\\ 0 & 1 & \frac12-\frac32 i\end{bmatrix} \xrightarrow[\text{(too tedious)}]{\text{Maple}} \overset{\text{LLF}}{\begin{bmatrix}1 & 0 & \frac12+\frac32 i\\ 0 & 1 & \frac12-\frac32 i\\ 0 & 0 & 0\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{aligned}x_1 &= -\left(\tfrac12+\tfrac32 i\right)t\\ x_2 &= -\left(\tfrac12-\tfrac32 i\right)t\\ x_3 &= t\end{aligned} \qquad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = t\begin{bmatrix}-\frac12-\frac32 i\\ -\frac12+\frac32 i\\ 1\end{bmatrix}$$

$$B = \begin{bmatrix}1 & -\frac12(1+3i) & -\frac12(1-3i)\\ 5/2 & -\frac12(1-3i) & -\frac12(1+3i)\\ 1 & 1 & 1\end{bmatrix}$$

$$\vec x = B\vec y = y_1\vec b_1 + y_2\vec b_2 + y_3\vec b_3$$

$$\vec y\,' = A_D\vec y = \begin{bmatrix}0 & 0 & 0\\ 0 & -\frac35+\frac{3}{10}i & 0\\ 0 & 0 & -3/5-\frac{3}{10}i\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} \qquad \begin{aligned}y_1' &= 0\\ y_2' &= \left(-\tfrac35+\tfrac{3}{10}i\right)y_2\\ y_3' &= \left(-\tfrac35-\tfrac{3}{10}i\right)y_3\end{aligned}$$

$$\begin{aligned}y_1 &= c_1\\ y_2 &= c_2 e^{-\frac35 t}e^{\frac{3i}{10}t}\\ y_3 &= c_3 e^{-3/5 t}e^{\frac{3it}{10}}\end{aligned}$$

b) continued

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = c_1\begin{bmatrix}1\\5/2\\1\end{bmatrix} + c_2 e^{-\frac35 t}e^{\frac{3it}{10}}\begin{bmatrix}-\frac12(1+3i)\\ -\frac12(1-3i)\\ 1\end{bmatrix} + c.c.$$

$$e^{-\frac35 t}\left(\cos\tfrac{3t}{10} + i\sin\tfrac{3t}{10}\right)\begin{bmatrix}-\frac12(1+3i)\\ -\frac12(1-3i)\\ 1\end{bmatrix}$$

$$= e^{-3/5t}\begin{bmatrix}-\frac12(c-3s+i(s+3c))\\ -\frac12(c+3s+i(s-3c))\\ c+is\end{bmatrix}$$

$$= e^{-3t/5}\begin{bmatrix}-\frac12\left(\cos\frac{3t}{10}-3\sin\frac{3t}{10}\right)\\ -\frac12\left(\cos\frac{3t}{10}+3\sin\frac{3t}{10}\right)\\ \cos\frac{3t}{10}\end{bmatrix} + i e^{-3t/5}\begin{bmatrix}-\frac12\left(\sin\frac{3}{10}t+3\cos\frac{3t}{10}\right)\\ -\frac12\left(\sin\frac{3}{10}t-3\cos\frac{3t}{10}\right)\\ \sin\frac{3t}{10}\end{bmatrix}$$

$$\equiv \vec X_2 + i\vec X_3 \to \text{real basis } \{\vec X_2, \vec X_3\}$$

$$\boxed{\vec x = c_1\vec b_1 + c_2\vec X_2 + c_3\vec X_3} \quad \text{gen soln}$$

c)

$$\vec x(0) = c_1\vec b_1 + c_2\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix} + c_3\begin{bmatrix}-3/2\\3/2\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1/2 & -3/2\\ 5/2 & -1/2 & 3/2\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\9\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}\ \end{bmatrix}^{-1}\begin{bmatrix}0\\9\\0\end{bmatrix} \overset{\text{maple}}{=} \begin{bmatrix}2\\-2\\2\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 2\begin{bmatrix}1\\5/2\\2\end{bmatrix} + e^{-\frac{3t}{5}}\begin{bmatrix}+(c-3s)\\+(c+3s)\\-2c\end{bmatrix} + e^{-\frac{3t}{5}}\begin{bmatrix}-(s+3c)\\-(s-3c)\\2s\end{bmatrix}$$

$$= \boxed{\begin{bmatrix}2+e^{-\frac{3t}{5}}(-2c-4s)\\ 5+e^{-\frac{3t}{5}}(4c+2s)\\ 2+e^{-\frac{3t}{5}}(-2c+2s)\end{bmatrix}} \qquad \begin{aligned}C &\equiv \cos\tfrac{3t}{10}\\ S &\equiv \sin\tfrac{3t}{10}\end{aligned}$$

maple agrees. ✓

d) $\boxed{\lim_{t\to\infty}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2\\5\\2\end{bmatrix}}$

e) $\tau = 5/3$

$5\tau = \frac{25}{3} \sim \underline{\underline{10}}$

viewing window

$t = 0..10$

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(3) a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} -10 & 2 \\ 3 & -15 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$

$0 = |A - \lambda I| = \begin{vmatrix} -10-\lambda & 2 \\ 3 & -5-\lambda \end{vmatrix} = (\lambda+5)(\lambda+10) - 6$

$= \lambda^2 + 15\lambda + 50 - 6 = \lambda^2 + 15\lambda + 44 = 0$

$\xrightarrow{\text{maple}} \quad \lambda = -9, -16$

$\lambda = -9: \quad A + 9I = \begin{bmatrix} -10+9 & 2 \\ 3 & -15+9 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

$\xrightarrow{\text{LF}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 = 2t \\ x_2 = t \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{b}_1}$

$\lambda = -16: \quad A + 16I = \begin{bmatrix} -10+16 & 2 \\ 3 & -15+16 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & +1 \end{bmatrix}$

$\xrightarrow{\text{LF}} \begin{bmatrix} 1 & +1/3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 = -\frac{1}{3}t \\ x_2 = t \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$

rescale: $\vec{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\boxed{\begin{matrix} \lambda = & -9 & -16 \\ B = & \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \\ & \vec{b}_1 & \vec{b}_2 \end{matrix}}$ $\quad B^{-1} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \quad A_D = B^{-1} A B = \begin{bmatrix} -9 & 0 \\ 0 & -16 \end{bmatrix}$

c) $\vec{x} = B\vec{y}, \quad \vec{y} = B^{-1}\vec{x}: \quad \vec{x}' = A\vec{x} \rightarrow$

$\vec{y}' = A_D y \rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -9y_1 \\ -16y_2 \end{bmatrix}$

$y_1' = -9y_1 \qquad y_1 = c_1 e^{-9t}$

$y_2' = -16y_2 \qquad y_2 = c_2 e^{-16t}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1\vec{b}_1 + y_2\vec{b}_2 = c_1 e^{-9t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-16t} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

e) $\tau_1 = 1/9 \quad > \quad \tau_2 = \frac{1}{16}$

$5\tau_1 = 5/9$ decay window. ≈ 0.51

b) $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$ new coords.

$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = B^{-1} \begin{bmatrix} 8 \\ -3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 24-3 \\ -8-6 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ -2 \end{bmatrix}}$

$\begin{bmatrix} 8 \\ -3 \end{bmatrix} = 3\vec{b}_1 - 2\vec{b}_2$

c) continued $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3e^{-9t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2e^{-16t} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 6e^{-9t} + 2e^{-16t} \\ 3e^{-9t} - 6e^{-16t} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$

IVP soln

e) $x_2 = 3e^{-9t} - 6e^{-16t}$

$[x_2' = -27e^{-9t} + 6 \cdot 16 e^{-16t} = 0] \, e^{16t}$

$-9e^{7t} + 32 = 0$

$e^{7t} = 32/9 \qquad \boxed{t_{max} = \frac{1}{7} \ln 32/9 \approx 0.181}$

$x_2(t_{max}) = 3e^{-\frac{9}{7}\ln\frac{32}{9}} - 6e^{-\frac{16}{7}\ln 32/9}$

$= 3\left(\frac{9}{32}\right)^{9/7} - 6\left(\frac{9}{32}\right)^{\frac{16}{7}} \xrightarrow{\text{maple}}$

$= \frac{189}{16384} 32^{5/7} 9^{2/7} \approx \boxed{0.257}$

(0.18, 0.257) pt on graph looks right.