MAT2705-03/04 101S Quiz 4 Print Name (Last, First) ____________________|

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

A population model is described by the initial value problem $\frac{dP}{dt} = 2\,e^{-3t} P$, $P(0) = P_0$.

a) What is the characteristic time τ of the exponential decay factor in the differential equation?

b) Find the general solution of the differential equation. ~~Combine into a single exponential~~. oops!

c) Find the solution which satisfies the initial condition. Combine into a single exponential. Does it agree with Maple (when "Simplified")?

d) Evaluate the limiting population $P_\infty = \lim_{t \to \infty} P(t)$, exactly and numerically to 2 decimal places.

e) At what value T of t does the population rise to 95 percent of the limiting population? Give its exact and numerical value to 2 decimal places, using rules of exponents and logs to isolate t. How many multiples of the characteristic time does this represent?

f) Setting $P_0 = 1$, equivalent to measuring the population in units of the limiting population, make a rough sketch of your solution together with the constant functions P_∞ and $0.95\ P_\infty$ in the window $t = 0..1.5$, $P = 0..3$. Mark the crossing point of the solution with the 95 percent line.

g) Estimate roughly the crossing point value of t from your plot. Does this agree with part e)? oops!

h) **Optional.** Write down the solution to $\frac{dP}{dt} = 2\,P$, $P(0) = P_0$. Include its graph in your previous diagram with $P_0 = 1$. For what fraction of the characteristic time τ do the two solutions remain roughly comparable before the unchecked growth solution pulls away?

To check with Maple, you need to use function notation: $P(t)$ and $P'(t)$ in the differential equation.

solution

a) $e^{-3t} \to k=3$, $\boxed{\tau = \frac{1}{k} = \frac{1}{3} \approx 0.33}$

b) $\frac{dP}{dt} = 2e^{-3t} P$ separable (and linear)

$\int \frac{dP}{P} = \int 2e^{-3t}\,dt$ separate & integrate

$\ln P = -\frac{2}{3}e^{-3t} + C_1$ $(P \geq 0)$

$e^{\ln P} = e^{-\frac{2}{3}e^{-3t} + C_1}$

$\boxed{P = \underbrace{e^{C_1}}_{C} e^{-\frac{2}{3}e^{-3t}} = C\,e^{-\frac{2}{3}e^{-3t}}}$

general soln

c) $P_0 = P(0) = C e^{-2/3} \to C = P_0 e^{2/3}$

$\boxed{P = P_0 e^{2/3} e^{-2/3 e^{-3t}} = P_0 e^{\frac{2}{3} - \frac{2}{3}e^{-3t}} = P_0 e^{\frac{2}{3}(1-e^{-3t})}}$ IVP soln

Agrees with Maple when "Simplified"

d) $P_\infty = \lim_{t\to\infty} P_0 e^{2/3(1-\underbrace{e^{-3t}}_{\to 0})} = \boxed{P_0 e^{2/3} \approx 1.95\,P_0}$

e) $.95 P_0 e^{2/3} = P_0 e^{2/3} e^{-\frac{2}{3}e^{-3t}}$

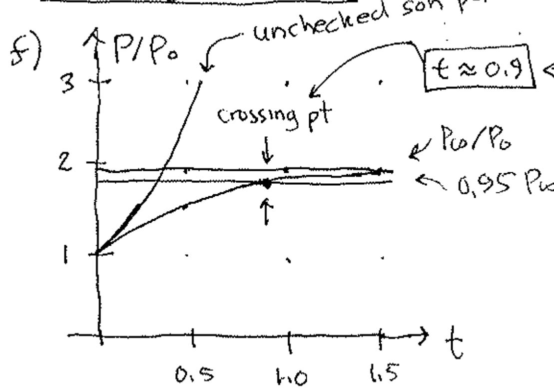
$.95 = e^{-\frac{2}{3}e^{-3t}} \to \ln 0.95 = -\frac{2}{3}e^{-3t}$ top of next column

e) continued. $e^{-3t} = -\frac{3}{2}\ln 0.95$

$\boxed{t = -\frac{1}{3}\ln\left(-\frac{3}{2}\ln 0.95\right) \approx 0.85}$

$\boxed{\frac{t}{\tau} = \frac{t}{1/3} \approx 2.56}$

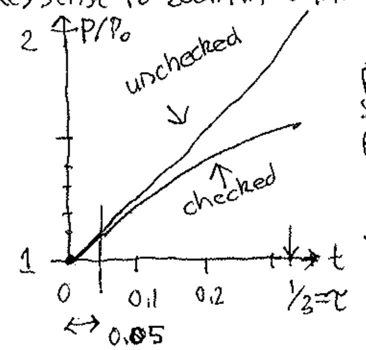
f)



g) 0.9 is pretty close to 0.85, so yes they agree

$P = P_0 e^{2t}$ is the unchecked soln.

h) Makes sense to zoom in to interval of interest



pixels begin to separate at $t \approx 0.05$

$\frac{t}{\tau} \approx \frac{0.05}{1/3} \approx 0.15$

$\boxed{15\% \text{ of } \tau}$