① a) $4x''+4x'+17x=F(t)$

$x''+x'+\frac{17}{4}x = F(t)/4$

$k_0=1$ $\quad \omega_0^2=17/4$

$\tau_0=1/k_0=1 \quad \omega_0=\sqrt{17}/2\approx 2.062$

$Q=\omega_0\tau_0=\sqrt{17}/2\approx 2.062$

$T_0=2\pi/\omega_0 = 4\pi/\sqrt{17}\approx 3.048$

b) $4x''+4x'+17x=0$

$x=e^{rt} \to \underbrace{(4r^2+4r+17)}_{=0}e^{rt}=0$

$r=\frac{-4\pm\sqrt{16-4(4)17}}{2\cdot 4}=-\frac{1\pm\sqrt{-16}}{2}=\frac{-1\pm 4i}{2} = -\frac{1}{2}\pm 2i$

$e^{rt}=e^{-\left(\frac{1\pm 4i}{2}\right)t}=e^{-t/2}(\cos 2t\pm i\sin 2t)$

$\hookrightarrow$ real basis: $e^{-t/2}\cos 2t,\ e^{-t/2}\sin 2t$

$x= e^{-t/2}(c_1\cos 2t+c_2\sin 2t)$

$\tau_1=2, \quad \omega_1=2, \ T_1=2\pi/2=\pi\approx 3.142$

c) $4x''+4x'+17x=-4\sin 2t$

$17[x_p= c_3\cos 2t+c_4\sin 2t\ ]$
$4[x_p'=-2c_3\sin 2t+2c_4\cos 2t]$
$4[x_p''=-4c_3\cos 2t-4c_4\sin 2t]$

$4x_p''+4x_p'+17x_p=[17-4(4)]c_3+4(2)c_4]\cos 2t + [-4(2)c_3+[17-4(4)]c_4]\sin 2t$

$=\underbrace{(c_3+8c_4)}_{=0}\cos 2t+\underbrace{(-8c_3+c_4)}_{=-4}\sin 2t = -4\sin 2t$

$\begin{bmatrix}1&8\\-8&1\end{bmatrix}\begin{bmatrix}c_3\\c_4\end{bmatrix}=\begin{bmatrix}0\\-4\end{bmatrix}, \quad \begin{bmatrix}c_3\\c_4\end{bmatrix}=\frac{1}{1+64}\begin{bmatrix}1&-8\\8&1\end{bmatrix}\begin{bmatrix}0\\-4\end{bmatrix}=\frac{4}{65}\begin{bmatrix}8\\-1\end{bmatrix}$

$x_p=(4/65)(8\cos 2t-\sin 2t) = x_{ss}$ steady state soln

$x_h= e^{-t/2}(c_1\cos 2t+c_2\sin 2t) \leftarrow$ part b)

$x=e^{-t/2}(c_1\cos 2t+c_2\sin 2t)+\frac{4}{65}(8\cos 2t-\sin 2t)$

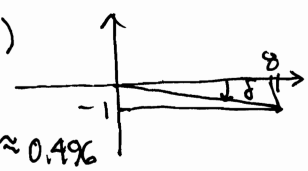
$x'=-\frac{1}{2}e^{-t/2}(c_1\cos 2t+c_2\sin 2t)+\frac{4(2)}{65}(-8\sin 2t-\cos 2t) + e^{-t/2}(-2\sin 2t+2c_2\cos 2t)$

$0=x(0)= c_1+32/65 \to c_1=-32/65$

$0=x'(0)=-\frac{1}{2}c_1+2c_2-8/65 \to c_2=\frac{1}{2}\left(\frac{8}{65}+\frac{1}{2}\left(\frac{-32}{65}\right)\right)=-\frac{4}{65}$

$$\boxed{x=\frac{1}{65}\left[e^{-t/2}(-32\cos 2t-4\sin 2t)+32\cos 2t-4\sin 2t\right]}$$

$x_{ss}=\frac{4}{65}(8\cos 2t-\sin 2t)$



$\delta=-\arctan 1/8$

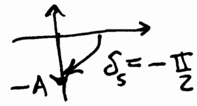
$A=\frac{4}{65}\sqrt{1+64}=\frac{4}{\sqrt{65}}\approx 0.496$

c) continued. $\delta=-\arctan 1/8\approx -0.124$ radians
≈ -7.13 degrees
≈ -0.0198 cycles.

see plots.

d) $-A\sin 2t = 0\cos 2t - A\sin 2t$

$\hookrightarrow (0,-A)$:



$\delta-\delta_s=\frac{\pi}{2}-\arctan 1/8$
≈ 82.9 degrees
≈ 0.230 cycles

about 1/4 cycle behind (at a later time)

The plot exactly shows this lag of about 1/4 cycle. (steady state peaks to the right of the driving function peaks)

e) $4x''+4x'+17x=-4A_0\omega^2\sin\omega t$

$17[x_p= c_3\cos\omega t+c_4\sin\omega t\ ]$
$4[x_p'=-\omega c_3\sin\omega t+\omega c_4\cos\omega t]$
$4[x_p''=-\omega^2c_3\cos\omega t-\omega^2c_4\sin\omega t]$

$4x_p''+4x_p'+17x_p=[(17-4\omega^2)c_3+4\omega c_4]\cos\omega t + [-4\omega c_3+(17-4\omega^2)c_4]\sin\omega t = -4A_0\omega^2\sin\omega t$

$\begin{bmatrix}17-4\omega^2 & 4\omega\\ -4\omega & 17-4\omega^2\end{bmatrix}\begin{bmatrix}c_3\\c_4\end{bmatrix}=\begin{bmatrix}0\\-4A_0\omega^2\end{bmatrix}$

$\begin{bmatrix}c_3\\c_4\end{bmatrix}=\frac{1}{(17-4\omega^2)^2+16\omega^2}\begin{bmatrix}17-4\omega^2 & -4\omega\\ 4\omega & 17-4\omega^2\end{bmatrix}\begin{bmatrix}0\\-4A_0\omega^2\end{bmatrix}$

$=\frac{+4A_0\omega^2}{(17-4\omega^2)^2+16\omega^2}\begin{bmatrix}4\omega\\-(17-4\omega^2)\end{bmatrix}$

$x_p=\frac{4A_0\omega^2}{(17-4\omega^2)^2+16\omega^2}\left[4\omega\cos\omega t-(17-4\omega^2)\sin\omega t\right]$

$\mathcal{D} \hookrightarrow = 17^2-8(17)\omega^2+16\omega^4+16\omega^2$
$= 289-120\omega^2+16\omega^4 \leftarrow$ in Maple soln.

f) $A(\omega)=\frac{4A_0\omega^2}{(17-4\omega^2)^2+16\omega^2}\left[16\omega^2+(17-4\omega^2)^2\right]^{1/2}$

$=\frac{4A_0\omega^2}{\sqrt{(17-4\omega^2)^2+16\omega^2}}\geq 0$

I used product rule (or use quotient rule)

$0=A'(\omega)=\frac{1}{2}[\mathcal{D}]^{-3/2}4A_0\omega^2(-240\omega+64\omega^3) + \frac{8A_0\omega}{\sqrt{\mathcal{D}}}=\frac{8A_0\omega}{\mathcal{D}^{3/2}}[\mathcal{D}+60\omega^2-16\omega^4]$

$289-120\omega^2+16\omega^4$

$=\frac{8A_0\omega}{\mathcal{D}^{3/2}}(289-60\omega^2)\to \boxed{\omega_p=\sqrt{\frac{289}{60}}=\frac{17}{2\sqrt{15}} \approx 2.194}$

(Note ω_p is slightly larger than $\omega_0\approx 2.062$.)

MAT 2500-03/04 [illegible]S TakeHomeTest 3 (2)

① f) continued.

$$A(\omega_p) = \frac{4A_0\left(\frac{17^2}{60}\right)}{\sqrt{\left(17 - \frac{4\cdot 17^2}{60_{15}}\right)^2 + 16\left(\frac{17^2}{60}\right)}} \quad \frac{4\cdot 17^2}{15^2}(1+15)$$

$$\left(\frac{17(15-17)}{15}\right)^2 = \frac{16\cdot 17^2}{15^2}$$

$$= \frac{4A_0\, 17^2}{60\sqrt{\frac{4\cdot 17^2\, 16}{15^2}}} = \frac{4A_0 17^{\not 2}\, 15}{60\cdot 2\cdot 17\cdot 4} = \frac{17}{8}A_0 \text{ whew.}$$

just using Maple is acceptable

$$\boxed{\frac{A(\omega_p)}{A_0} = \frac{17}{8} = 2.125} \Leftrightarrow Q \approx 2.062$$

hmm. they are comparable.

$$A(2) = \frac{4(2^{-2})2^2}{\sqrt{(17-16)^2 + 16\cdot 4}} = \boxed{\frac{4}{\sqrt{65}}}\,! \checkmark$$ yes, they agree.

g) $$q_\infty = \lim_{\omega\to\infty}\frac{A(\omega)}{\omega_0} = \lim_{\omega\to\infty}\frac{4\omega^2}{\sqrt{(17-4\omega^2)^2 + 16\omega^2}}$$

$$= \lim_{\omega\to\infty}\frac{4\omega^2}{4\omega^2\sqrt{\frac{(17-4\omega^2)^2}{16\omega^4} + \frac{16\omega^2}{16\omega^4}}}$$

$$= \lim_{\omega\to\infty}\frac{1}{\sqrt{\left(1 - \frac{17}{4\omega^2}\right)^2 + \frac{1}{\omega^2}}} = \boxed{1}$$ acceptable to use Maple.

$$x_p = \frac{4A_0\omega^2}{(17-4\omega^2)^2 + 16\omega^2}\left[4\omega\cos\omega t - (17-4\omega^2)\sin\omega t\right]$$ keep highest powers

$$\sim \frac{4A_0\omega^2}{16\omega^4}\left[4\omega\cos\omega t + 4\omega^2\sin\omega t\right] \quad \left(\frac{\omega^3}{\omega^4}\to 0\right)$$

$$\sim \boxed{A_0\sin\omega t} \text{ for } \omega >> 1$$

h) see plots. $$\frac{A(\omega_0)}{A_0} = \frac{4(17/4)}{\sqrt{(17-4(17/4))^2 + 16(17/4)}} = \frac{\sqrt{17}}{\sqrt{4}} = \frac{\sqrt{17}}{2} = Q$$

② a) $x_1' = -8x_1 - 5x_2$, $x_1(0) = -3$

$x_2' = 9x_1 + 4x_2$, $x_2(0) = 15$

b) $$0 = |A - \lambda I| = \begin{vmatrix} -8-\lambda & -5 \\ 9 & 4-\lambda\end{vmatrix} = (\lambda+8)(\lambda-4) + 45$$

$$= \lambda^2 + 4\lambda + 13 \to \lambda = \frac{-4 \pm \sqrt{16 - 4\cdot 13}}{2} = -2 \pm \sqrt{4-13}$$

$$= -2 \pm 3i$$

$$\lambda = -2+3i: \; A - \lambda I = \begin{bmatrix} -8+2-3i & -5 \\ 9 & 4+2-3i\end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & 2/3 - i/3 \\ 0 & 0\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \to x_1 = -\frac{(2-i)}{3}x_2$$

L F $x_2 = t$

$$\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = \begin{bmatrix} -\frac{(2-i)}{3}t \\ t\end{bmatrix} = t\begin{bmatrix} -(2-i)/3 \\ 1\end{bmatrix} \quad \vec{b}_1, \quad \vec{b}_2 = \overline{\vec{b}_1}$$

$$B = \begin{bmatrix} -(2-i)/3 & -(2+i)/3 \\ 1 & 1\end{bmatrix} \quad A_B = \begin{bmatrix} -2+3i & 0 \\ 0 & -2-3i\end{bmatrix}$$

$$\vec{x} = B\vec{y} \to \vec{x}\,' = A\vec{x} \to B^{-1}(B\vec{y})' = B^{-1}AB y$$

$$\vec{y}\,' = A_B y \to y_1' = (-2+3i)y_1 \quad y_1 = c_1 e^{(-2+3i)t}$$

$$y_2' = (-2-3i)y_2 \quad y_2 = c_2 e^{(-2-3i)t}$$

$$\vec{x} = y_1\vec{b}_1 + y_2\vec{b}_2 = c_1 e^{(-2+3i)t}\vec{b}_1 + c_2 e^{(-2-3i)t}\vec{b}_2$$

$$e^{-2t}(\cos 3t + i\sin 3t)\begin{bmatrix} -(2-i)/3 \\ 1\end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix} \frac{1}{3}\left[-2\cos 3t - \sin 3t + i(-2\sin 3t + \cos 3t)\right] \\ \cos 3t + i\sin 3t\end{bmatrix}$$

$$= \underbrace{e^{-2t}\begin{bmatrix} -\frac{1}{3}(2\cos 3t + \sin 3t) \\ \cos 3t\end{bmatrix}}_{} + i\underbrace{e^{-2t}\begin{bmatrix}\frac{1}{3}(\cos 3t - 2\sin 3t) \\ \sin 3t\end{bmatrix}}_{}$$

new basis of soln space (real!)

GENSOLN:

$$\boxed{\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} -\frac{1}{3}(2\cos 3t + \sin 3t) \\ \cos 3t\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}\frac{1}{3}(\cos 3t - 2\sin 3t) \\ \sin 3t\end{bmatrix}}$$

c) $$\begin{bmatrix} -3 \\ 15\end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0)\end{bmatrix} = c_1\begin{bmatrix} -2/3 \\ 1\end{bmatrix} + c_2\begin{bmatrix} 1/3 \\ 0\end{bmatrix} = \begin{bmatrix} (-2c_1 + c_2)/3 \\ c_1\end{bmatrix}$$

$c_1 = 15$, $c_2 = 2c_1 + 3(-3) = 2(15) - 9 = 21$

$$\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = e^{-2t}\begin{bmatrix} -\frac{\cancel{15}^{\,5}}{\cancel{3}}(2\cos 3t + \sin 3t) + \frac{\cancel{21}^{\,7}}{\cancel{3}}(\cos 3t - 2\sin 3t) \\ 15\cos 3t + 21\sin 3t\end{bmatrix}$$

$$\boxed{= \begin{bmatrix} e^{-2t}(-3\cos 3t - 19\sin 3t) \\ e^{-2t}(15\cos 3t + 21\sin 3t)\end{bmatrix}}$$

d) $A_{10} = \sqrt{3^2 + 19^2} = \sqrt{370} \approx 19.24$

$A_{20} = 3\sqrt{5^2 + 7^2} = 3\sqrt{74} \approx 25.81$

envelopes: $\pm A_{10}e^{-2t}$, $\pm A_{20}e^{-2t}$

e) $\tau = 1/2 \to 5\tau = 2.5$ plot $t = 0..2.5$ (see plots)

③ a) $$\begin{bmatrix} x_1 \\ x_2\end{bmatrix}' = \underbrace{\begin{bmatrix} -5 & -2 \\ -1 & -4\end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2\end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0)\end{bmatrix} = \begin{bmatrix} -4 \\ 1\end{bmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} -5-\lambda & -2 \\ -1 & -4-\lambda\end{vmatrix} = (\lambda+4)(\lambda+5) - 2$$

$$= \lambda^2 + 9\lambda + 18$$

$$\lambda = \frac{-9 \pm \sqrt{9^2 - 4\cdot 9\cdot 2}}{2} = \frac{-9 \pm 3\sqrt{9-8}}{2} = \frac{-9\pm 3}{2}$$

$$= -3, -6$$

(3) a) continued

$\lambda_1 = -3:\quad A+3I = \begin{bmatrix} -5+3 & -2 \\ -1 & -4+3 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{LF} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_2 = t, \; x_1 = -x_2 = -t \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{b}_1}$

$\lambda_2 = -6:\quad A+6I = \begin{bmatrix} -5+6 & -2 \\ -1 & -4+6 \end{bmatrix} = \begin{bmatrix} +1 & -2 \\ -1 & +2 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}}_{LF} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_2 = t, \; x_1 = 2x_2 = +2t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +2t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} +2 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

$B = \begin{bmatrix} -1 & +2 \\ 1 & 1 \end{bmatrix}, \quad A_B = B^{-1}AB = \begin{bmatrix} -3 & 0 \\ 0 & -6 \end{bmatrix}, \quad B^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$

b) $\vec{x} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} = B\vec{y} \to \vec{y} = B^{-1}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4+2 \\ -4+1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

so $\vec{x}(0) = 2\vec{b}_1 - \vec{b}_2$

d) The IVP solution is just $\vec{x} = 2e^{-3t}\vec{b}_1 - e^{-6t}\vec{b}_2 = 2e^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} - e^{-6t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2e^{-3t} - 2e^{-6t} \\ 2e^{-3t} - e^{-6t} \end{bmatrix}$$

$x_2 = 2e^{-3t} - e^{-6t}$

$x_2' = -6e^{-3t} + 6e^{-6t}$

$x_2'' = 18e^{-3t} - 36e^{-6t} = 9e^{-6t}(2e^{3t} - 4) = 0 \to e^{3t} = \frac{4}{2} = 2 \to t = \frac{1}{3}\ln 2 \approx 0.231$

$\hookrightarrow x_2 = 2e^{-3(\frac{1}{3}\ln 2)} - e^{-6(\frac{1}{3}\ln 2)} = 2e^{-\ln 2} - e^{-2\ln 2} = 2(2)^{-1} - (2)^{-2} = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$

point $(0.23, 0.75)$ on plot

A point of inflection on a graph is where the concavity changes direction (or the 2nd derivative changes sign). $x_2(t)$ is concave down at $t=0$ but clearly concave up for $t \geq 1$ so in between it must switch!