MAT2705-03/04 11S Take Home Test 3 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions, matrix inverses, plotting and root finding.** ***Print*** requested technology plots, annotate them appropriately by hand and attach to the relevant problems.

1. The displacement $x(t)$ of an underdamped harmonic oscillator system satisfies

$$m\,x''(t) + c\,x'(t) + K\,x(t) = F(t).$$

Let $m = 4$, $c = 4$, $K = 17$ and the initial conditions $x(0) = 0, x'(0) = 0$.

a) Express this DE in standard form with a unit coefficient of the second derivative term. What are the natural frequency ω_0, natural decay time τ_0, the quality factor $Q = \omega_0 \tau_0$ and the period $T_0 = 2\,\pi/\omega_0$ for this system? (Give both exact and numeric values to 3 decimal places.)

Consider the following driving force functions $F(t)$:

b) $F(t) = 0$.

Find the general solution of the differential equation. What is the frequency ω_1, decay time τ_1 and the period $T_1 = 2\,\pi/\omega_1$ for this decaying sinusoidal solution? (Give both exact and numeric values to 3 decimal places.)

c) $F(t) = -m \sin(2\,t)$.

Find the initial value problem solution by hand (but check with Maple!)

Evaluate the values of the amplitude A and phase shift δ for the steady state solution (the part of the solution which remains after the transient has died away) and express the phase shift in radians, degrees and cycles (divide radians by 2π). Make a single plot in an appropriate viewing window showing both the solution function and the steady state solution until they merge. In a separate plot for comparison with the driving sine function, plot both the steady state solution and $-A \sin(2\,t)$ (same amplitude as the steady state solution) to see how the peaks of the steady state solution compare to the peaks of the driving function.

d) What is the phase shift δ_s of the driving function $-A \sin(2\,t)$ relative to $\cos(2\,t)$? By what signed angle between $-\pi$ and π on the unit circle does one have to move from the angle δ_s to get to the angle δ? (namely, $\delta - \delta_s$ modulo an appropriate multiple of π) What fraction of the circle is this? Does your plot agree with this angle (does the steady state solution lead or lag behind the driving function by a corresponding amount)? Explain.

e) $F(t) = -m\,A_0\,\omega^2 \sin(\omega\,t), A_0 > 0$.

Explore resonance for this system by finding the steady state solution by hand, where the nonegative frequency ω of the driving force function is a parameter.

f) Evaluate the steady state amplitude function $A(\omega)$ and use calculus to find the exact and numerical value of the frequency ω_p and the amplitude $A(\omega_p)$ where it has its peak value for $\omega \geq 0$.

What is the numerical value of the ratio $A(\omega_p)/A_0$? How does it compare to the quality factor Q? Does $A(2)$ with $A_0 = 2^{-2}$ agree with your value for A from part c) as it should?

g) What is the limit of the amplitude ratio $a_\infty = \lim_{\omega \to \infty} A(\omega) / A_0$? What is the corresponding approximate steady state solution $x(t)$ for very large values of ω?

h) Plot the amplitude function ratio $A(\omega) / A_0$ in an appropriate window (showing the limiting behavior of the entire function for $\omega \geq 0$) together with the constant functions $A(\omega_0)/A_0$, $A(\omega_p)/A_0$ and a_∞ and hand annotate on your axes the values of these frequencies and amplitudes and indicate the points on the curve which correspond to ω_0 and ω_p.

2. $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -8 & -5 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 15 \end{bmatrix}$

a) Write this system of differential equations for the vector variable $\vec{x} = \langle x_1, x_2 \rangle$ AND its initial conditions in scalar form (use arrow notation for vectors when you write vectors!).
b) Use the eigenvector approach to find its general solution by hand, showing all steps.
c) Find the IVP solution, using matrix methods showing all steps. Make sure it agrees with Maple's solution.
d) Evaluate the upper and lower envelope functions for each of these damped sinusoidal functions by evaluating the amplitude of each sinusoidal factor (plus or minus amplitude times exponential factor).
e) Make a single plot showing the original two solution curves and their envelope curves versus t on the same axes for 5 characteristic times of the decaying exponential factor in these expressions (i.e., in an appropriate viewing window). Hand annotate each solution curve with its variable name.

3. $x_1'(t) = -5\,x_1(t) - 2\,x_2(t), x_2'(t) = -x_1(t) - 4\,x_2(t), x_1(0) = -4, x_2(0) = 1.$

a) Identify the coefficient matrix and find a new basis for R^2 consisting of eigenvectors $\vec{b}_1, \vec{b}_2$ of this matrix using the standard hand recipe. Order the real eigenvalues $\lambda_1 \geq \lambda_2$ by decreasing value.
b) Evaluate the new coordinates $\langle y_1, y_2 \rangle$ of the point $\langle x_1, x_2 \rangle = \langle -4, 1 \rangle$ with respect to this basis of eigenvectors.
c) Use technology to plot a directionfield for this DE with the solution curve through the single initial data point, and (by hand if necessary) include the lines through the two eigenvectors representing the two subspaces of eigenvectors. Choose the window $x_1 = -5..5, x_2 = -5..5$. By hand label these lines by their new coordinate labels, draw in and label the eigenvectors and the initial data vector $\vec{x}(0)$ themselves as arrows, and include the parallelogram projection of the latter vector onto the new coordinate axes, i.e., the parallelogram parallel to the new coordinate axes with the initial data vector as the main diagonal. Do the projections along the coordinate axes agree with the values you found for the new coordinates of this vector? Explain. Does your directionfield correspond to the eigenvectors you have drawn? Explain why.
d) Plot the two variables versus t for an appropriate viewing window for this initial value problem based on the longest characteristic time. Explain your window choice.The second variable x_2 has a point of inflection obvious in your plot. Find its coordinates and the corresponding value of t exactly and approximately. Annotate your diagram to show this point.

Advice. When in doubt about how much work to show, show more. Explain using words if it helps. Think of this take-home test as an exercise in "writing intensive" technical expression. Try to impress bob as though it were material for a job interview (you're fired! or you're hired! ?). In a real world technical job, you need to be able to write coherent technical reports that other people can follow.

▶ solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:
"During this examination, all work has been my own. I have read the long instructions on the class web page. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date: