MAT 2705-03/04 11S Test 1 Answers

① a) $y(0) \approx 1.22$ { arrows are 0.2 apart, my curve hits axis about 1/10 separation above 1.2 }

b) $2(y-1)^2 + (x-2)^{-1}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -2(y-1)^2(x-2)$ separable

now

$\int (y-1)^{-2}\,dy = \int -2(x-2)\,dx \quad (y \neq 1)$

$\frac{(y-1)^{-1}}{-1} = -(x-2)^2 + C$

$\frac{1}{y-1} = (x-2)^2 - C$

$y-1 = \frac{1}{(x-2)^2 - C}$

$$\boxed{y = 1 + \frac{1}{(x-2)^2 - C} = \frac{(x-2)^2 - C + 1}{(x-2)^2 - C} = \frac{x^2-4x+5-C}{x^2-4x+4-C}}$$

all 3 forms acceptable

c) $2 = y(1) = \frac{1-C+1}{1-C} = \frac{2-C}{1-C}$

$2(1-C) = 2-C$

$2-2C = 2-C \to 0 = C$

$$\boxed{y = 1 + \frac{1}{(x-2)^2} = \frac{x^2-4x+5}{x^2-4x+4}}$$

either acceptable

d) $y(0) = \frac{0+5}{0+4} = \frac{5}{4} = \boxed{1.25} \leftrightarrow 1.22$ hand drawn

not bad, close enough

e) Maple gives:

$y = \frac{-4x+5+x^2}{(x-2)^2}$ agreeing with the last expression above once you expand the denominator

f) y cannot equal 1 in this formula, but $y=1$ is an obvious isocline soln which satisfies original DE, and is obvious in the graph!

② a) $\frac{1}{t}\frac{dx}{dt} = e^{t^2/2} - x \to \frac{dx}{dt} = \underbrace{-tx + te^{t^2/2}}_{\text{linear in } x}$

standard linear form:

$e^{t^2/2}\left[\frac{dx}{dt} + t\,x = te^{t^2/2}\right]$

$e^{\int t\,dt} = e^{t^2/2}$

$\frac{d}{dt}\left(xe^{t^2/2}\right) = te^{t^2/2}e^{t^2/2} = te^{t^2}$

$\left[xe^{t^2/2} = \int \underbrace{e^{t^2}}_{e^u}\underbrace{t\,dt}_{du/2} = \frac{1}{2}e^{t^2} + C\right]e^{-t^2/2}$

$$\boxed{x = \left[\tfrac{1}{2}e^{t^2/2} + C\right]e^{-t^2/2} = \tfrac{1}{2}e^{t^2/2} + Ce^{-t^2/2}}$$

(multiply out) gen. soln.

b) $1 = x(0) = \frac{1}{2} + C \to C = \frac{1}{2}$

$$\boxed{x = \tfrac{1}{2}e^{t^2/2} + \tfrac{1}{2}e^{-t^2/2}}$$

IVP soln.

c) $x' = \frac{1}{2}e^{t^2/2}\,t + \frac{1}{2}e^{-t^2/2}(-t)$ $\frac{d}{dt}$

$\frac{x'}{t} = \frac{1}{2}\left(e^{t^2/2} - e^{-t^2/2}\right)$

$x^2 - \left(\frac{x'}{t}\right)^2 = \frac{1}{4}\left(e^{t^2/2} + e^{-t^2/2}\right)^2 - \frac{1}{4}\left(e^{t^2/2} - e^{-t^2/2}\right)^2$ eliminate unknown everywhere then simplify

$= \frac{1}{4}\left[e^{t^2} + e^{-t^2} + 2 - (e^{t^2} + e^{-t^2} - 2)\right]$

$= \frac{1}{4}[2+2] = 1$ ✓

③ a) $\frac{M-P}{P} = Ce^{-.04t}$ $\to$ $\boxed{\tau = \frac{1}{0.04} = 25}$

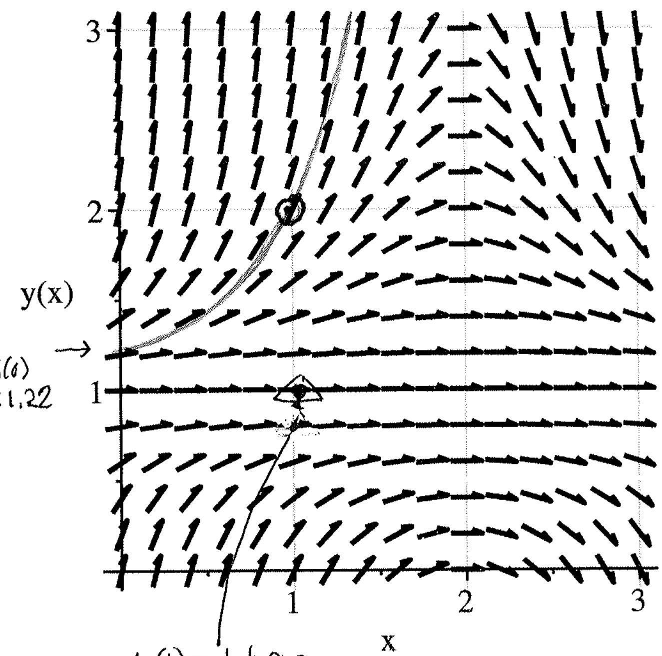
b) $\frac{19}{1} = Ce^{-.04t}\Big|_{t=0} = C \to \frac{M-P}{P} = 19e^{-.04t} \overset{\text{set equal}}{=} \frac{1}{19}$ solve for t

$19^2e^{-.04t} = 1$, $e^{.04t} = 19^2$, $t = \underbrace{\frac{1}{.04}}_{\tau}\ln 19^2 = \boxed{50\ln 19 \approx 147.22}$

$$\boxed{\frac{t}{\tau} = \ln 19^2 = 2\ln 19 \approx 5.89}$$

(so 90% of the S-curve spread vertically occurs within about 6 characteristic times)

optional: 1:19 → 19:1 goes to 1:99 → 99:1 so just replace 19 by 99: $\frac{t}{\tau} = \ln 99^2 \approx \boxed{9.19}$ ↔ 98% of rise



$y(0)$
≈ 1.22

tickmark separation = 1/5 = 0.2

y(1)=1 here —
isocline soln : y(x)=1