MAT 2705-03/04 11F Final Exam Answers (1)

a) $$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -7 & -6 \\ -1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 20\cos t \end{bmatrix} = \begin{bmatrix} -7x_1-6x_2 \\ -x_1-6x_2+20\cos t \end{bmatrix}$$

$$\boxed{\begin{array}{l} x_1'' = -7x_1-6x_2,\ x_2'' = -x_1-6x_2+20\cos t \\ x_1(0)=0,\ x_2(0)=0,\ x_1'(0)=0,\ x_2'(0)=0 \end{array}}$$

Maple soln: $$\boxed{\begin{array}{l} x_1 = -5\cos t + 8\cos 2t - 3\cos 3t \\ x_2 = 5\cos t - 4\cos 2t - \cos 3t \end{array}}$$

Note: $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5\cos t\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 4\cos 2t\begin{bmatrix} -2 \\ 1 \end{bmatrix} - \cos 3t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

b) $$0 = |A-\lambda I| = \begin{vmatrix} -7-\lambda & -6 \\ -1 & -6-\lambda \end{vmatrix} = (-7-\lambda)(-6-\lambda)-6 = (\lambda+7)(\lambda+6)-6$$

$$= \lambda^2+13\lambda+42-6 = \lambda^2+13\lambda+36 = 0 \rightarrow$$

$$\lambda = \frac{-13\pm\sqrt{169-4(36)}}{2} = \frac{-13\pm\sqrt{25}}{2} = \frac{-13\pm5}{2} = -4, -9$$

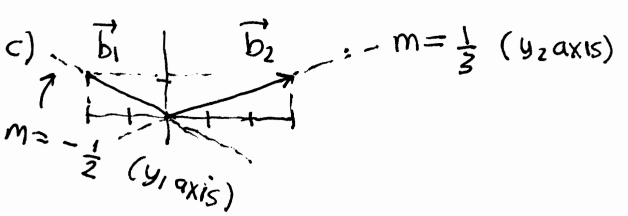
$$\lambda_1=-4:\ A+4I = \begin{bmatrix} -7+4 & -6 \\ -1 & -6+4 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{cc} L & F \end{array} \\ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \end{array}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2=t,\ x_1+2x_2=0 \rightarrow x_1=-2t \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec b_1}$$

$$\lambda_2=-9:\ A+9I = \begin{bmatrix} -7+9 & -6 \\ -1 & -6+9 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{cc} L & F \end{array} \\ \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \end{array}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2=t,\ x_1-3x_2=0 \rightarrow x_1=3t \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\vec b_2}$$

$$\boxed{\begin{array}{l} B = \langle \vec b_1|\vec b_2\rangle = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix},\ B^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & -3 \\ -1 & -2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \\ A_B = B^{-1}AB = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix} \end{array}}$$

c) 

d) $$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 12\cos t \\ 8\cos t \end{bmatrix} = \begin{bmatrix} -4y_1+12\cos t \\ -9y_2+8\cos t \end{bmatrix}$$

$$B^{-1}\vec F = \frac{1}{5}\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 20\cos t = \begin{bmatrix} 12\cos t \\ 8\cos t \end{bmatrix}$$

$$\boxed{\begin{array}{l} y_1'' = -4y_1+12\cos t \\ y_2'' = -9y_2+8\cos t \end{array}} \text{ or } \boxed{\begin{array}{l} y_1''+4y_1 = 12\cos t \\ y_2''+9y_2 = 8\cos t \end{array}}$$

d) continued.

$$\boxed{\begin{array}{l} y_{1h} = c_1\cos 2t + c_2\sin 2t \\ y_{2h} = c_3\cos 3t + c_4\sin 3t \end{array}}$$

$$y_{1p} = c_5\cos t + c_6\sin t$$

$$y_{1p}'' = -c_5\cos t - c_6\sin t$$

$$y_{1p}''+4y_{1p} = (-1+4)c_5\cos t + (-1+4)c_6\sin t$$

$$= \underbrace{3c_5}_{=12}\cos t + \underbrace{3c_6}_{=0}\sin t = 12\cos t$$

$$c_5 = 4,\ c_6 = 0$$

$$\boxed{y_{1p} = 4\cos t}$$

$$y_{2p} = c_7\cos t + c_8\sin t$$

$$y_{2p}'' = -c_7\cos t - c_8\sin t$$

$$y_{2p}''+9y_2 = (-1+9)c_7\cos t + (-1+9)c_8\sin t$$

$$= \underbrace{8c_7}_{=8}\cos t + \underbrace{8c_8}_{=0}\sin t = 8\cos t$$

$$c_7 = 1,\ c_8 = 0$$

$$\boxed{y_{2p} = \cos t}$$

$$\boxed{\begin{array}{l} y_1 = c_1\cos 2t + c_2\sin 2t + 4\cos t \\ y_2 = c_3\cos 3t + c_4\sin 3t + \cos t \end{array}}$$

e) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B\begin{bmatrix} c_1\cos 2t + c_2\sin 2t + 4\cos t \\ c_3\cos 3t + c_4\sin 3t + \cos t \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = B\begin{bmatrix} -2c_1\sin 2t + 2c_2\cos 2t - 4\sin t \\ -3c_3\sin 3t + 3c_4\cos 3t - \sin t \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B\begin{bmatrix} c_1+4 \\ c_3+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} c_1+4=0 \\ c_3+1=0 \end{array}$$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = B\begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} c_1=-4 \\ c_3=-1 \end{array}$$

$$\hookrightarrow \begin{array}{l} 2c_2=0 \\ 3c_4=0 \end{array} \qquad c_2=0,\ c_4=0$$

$$y_1 = -4\cos 2t + 4\cos t,\ y_{1h} = -4\cos 2t$$

$$y_2 = -\cos 3t + \cos t,\ y_{2h} = -\cos 3t$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4\cos 2t + 4\cos t \\ -\cos 3t + \cos t \end{bmatrix}$$

$$= \begin{bmatrix} (8\cos 2t - 8\cos t) + (-3\cos 3t + 3\cos t) \\ (-4\cos 2t + 4\cos t) + (-\cos 3t + \cos t) \end{bmatrix}$$

$$= \begin{bmatrix} 8\cos 2t - 3\cos 3t - 5\cos t \\ -4\cos 2t - \cos 3t + 5\cos t \end{bmatrix}$$

$$\boxed{\begin{array}{l} x_1 = 8\cos 2t - 3\cos 3t - 5\cos t \\ x_2 = -4\cos 2t - \cos 3t + 5\cos t \end{array}}$$ agrees with Maple!

MAT 2705-03/04 11F Final Exam Answers (2)

f) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{-4\cos 2t \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{b}_1} - \cos 3t \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\vec{b}_2}}_{\vec{x}_h} + \underbrace{\cos t \underbrace{\begin{bmatrix} -5 \\ 5 \end{bmatrix}}_{\vec{b}_3}}_{\vec{x}_p}$$

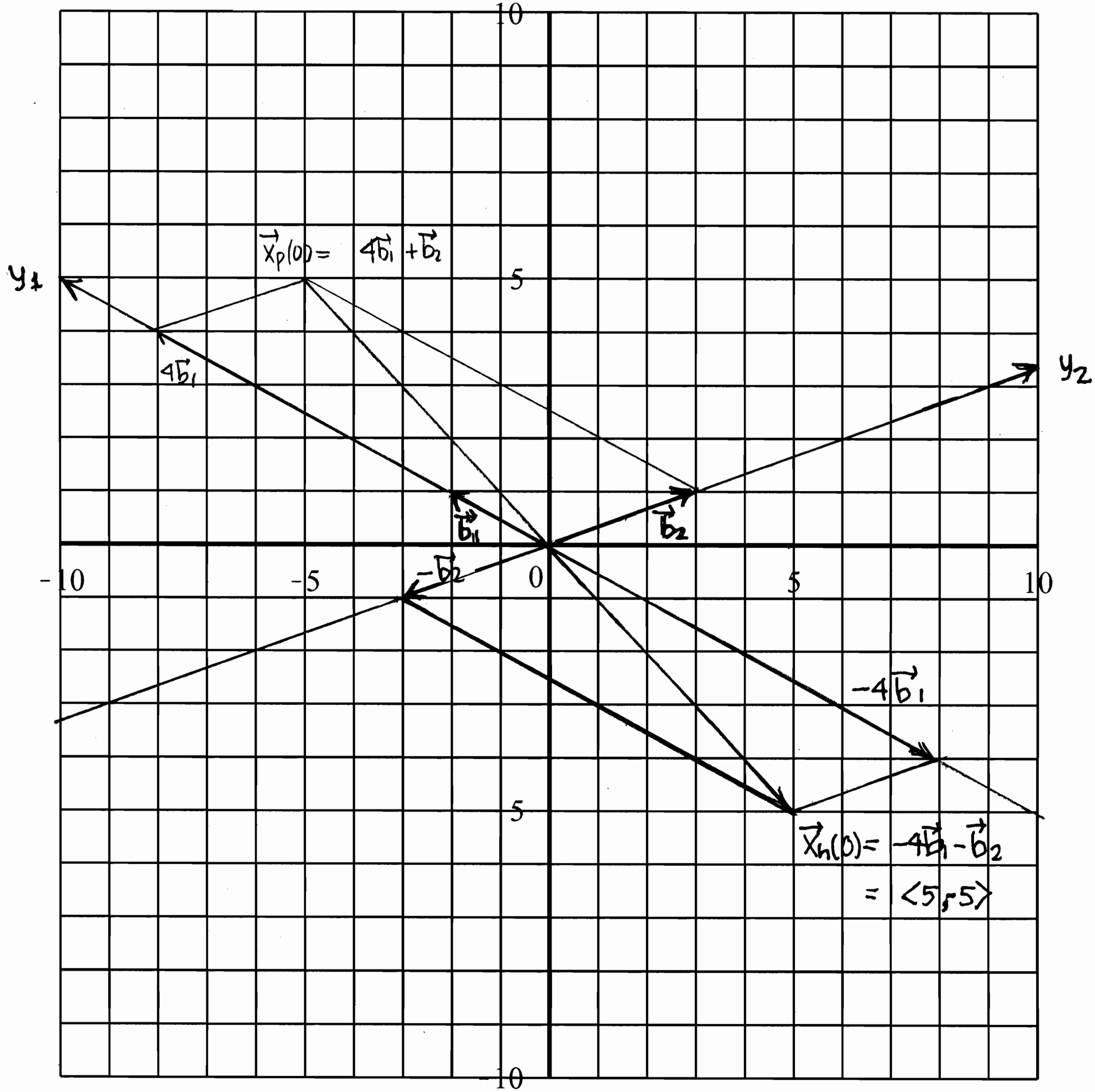
$\boxed{\begin{array}{c} \omega_1 = 2 \\ \text{accordian} \\ \text{mode} \end{array}}$, $\boxed{\begin{array}{c} \omega_2 = 3 \\ \text{tandem} \\ \text{mode} \end{array}}$

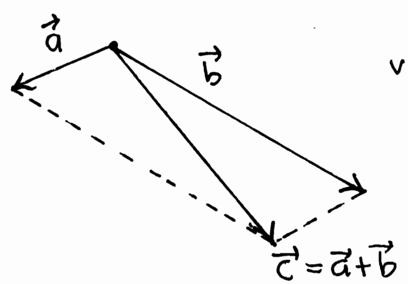
$\frac{x_1}{x_2} = -2 < 0$ $\qquad$ $\frac{x_1}{x_2} = 3 > 0$

g) $$\vec{x}_h(0) = -4\vec{b}_1 - \vec{b}_2 = \begin{bmatrix} -4(-2)-3 \\ -4-1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$(= -\vec{x}_p(0)\ !)$

$y_{1h}(0) = -4$, $y_{2h}(0) = -1$ new coords.





vector addition diagram